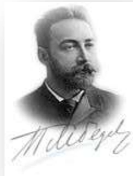

Some problems of extreme statistics of random walks and random operators

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Problems considered in the talk

1. Statistics of two-dimensional "stretched" random paths over a semicircle

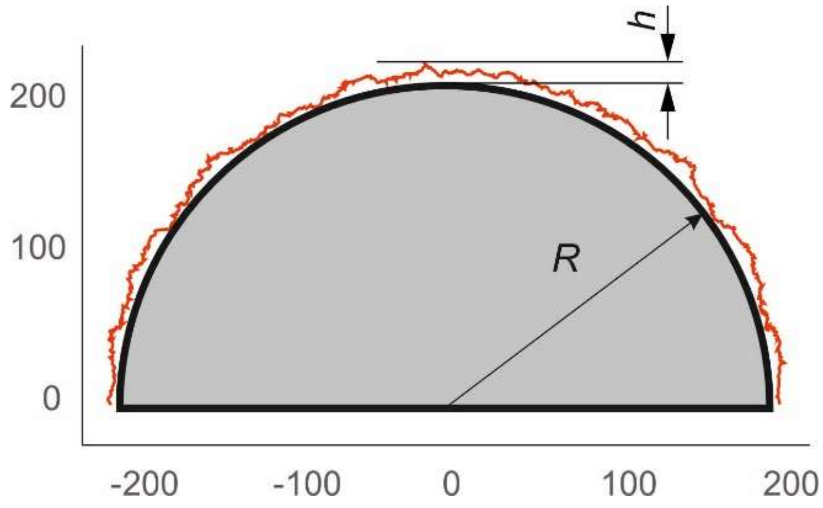
2. Statistics of one-dimensional random walks in a Poissonian field of traps

3. Spectral properties of random operators

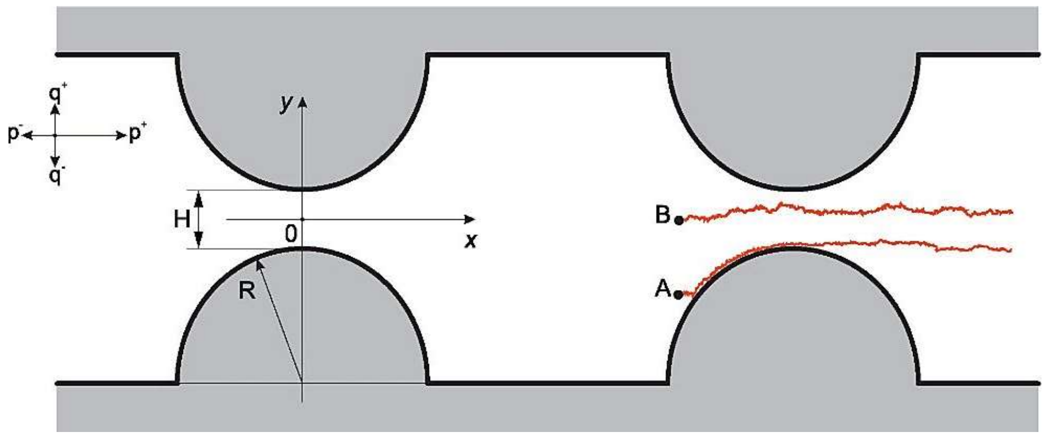
4. Connection with phyllotaxis

1. Statistics of two-dimensional "stretched" random paths over a semicircle

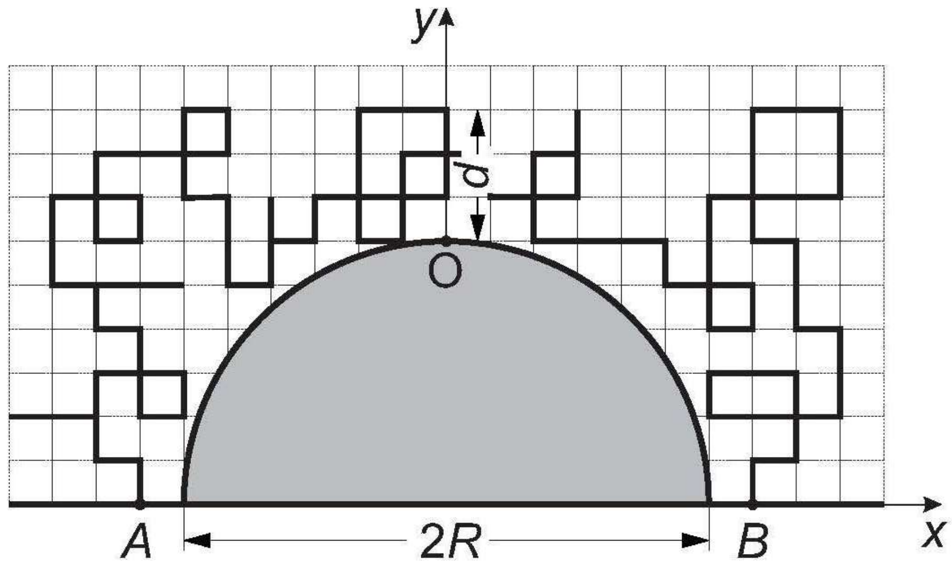
- Anomalous behavior of "elongated" two-dimensional random walks near convex boundaries



- Random walks with longitudinal drift in a two-dimensional non-uniform channel



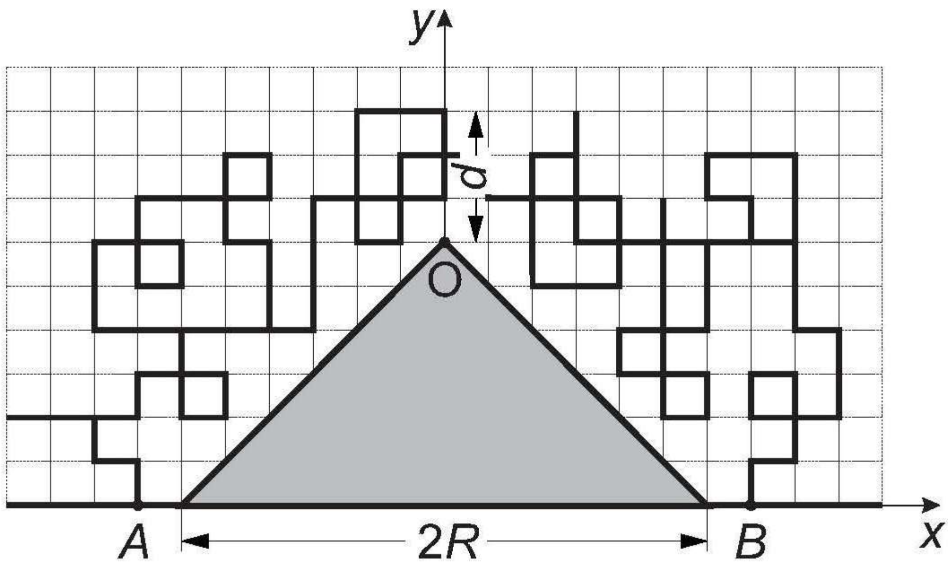
2D Random paths above voids of various shapes



Semicircle:

$$N \sim R^2 \;\rightarrow\; D \sim \sqrt{N}$$

$$N = cR \;\rightarrow\; D \sim N^{1/3} \sim R^{1/3}$$



Triangle:

$$N \sim R^2 \;\rightarrow D \sim \sqrt{N}$$

$$N = cR \;\rightarrow D \sim \text{const}$$

$$\begin{cases} \partial_N P_N(\rho,\phi|\rho_A,\phi_A) = D\Delta_{\rho,\phi} P_N(\rho,\phi|\rho_A,\phi_A) \\ P_N(\rho=R,\phi|\rho_A,\phi_A)) = P_N(\infty,\phi|\rho_A,\phi_A) = P_N(\rho,0|\rho_A,\phi_A) = P_N(\rho,2\pi|\rho_A,\phi_A) = 0 \\ P_N(\rho,\phi|\rho_A,\phi_A) = \frac{1}{\rho}\delta(\rho-\rho_A)\delta(\phi-\phi_A) \end{cases}$$

$$P_N(\rho,\phi|\rho_A,\phi_A) = \frac{2}{\pi}\sum_{k=1}^{\infty} \sin k\phi_A \sin k\phi \int_0^{\infty} \exp(-\lambda^2 DN) Z_k(\lambda\rho,\lambda R) Z_k(\lambda\rho_A,\lambda R)\lambda d\lambda$$

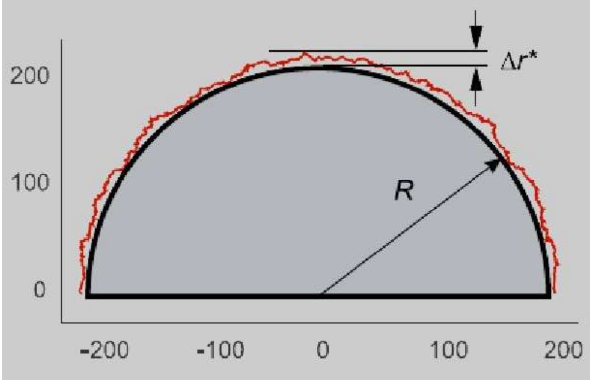
$$Z_k(\lambda\rho,\lambda R) = \frac{-J_k(\lambda\rho)N_k(\lambda R) + J_k(\lambda R)N_k(\lambda\rho)}{\sqrt{J_k^2(\lambda R) + N_k^2(\lambda R)}}$$

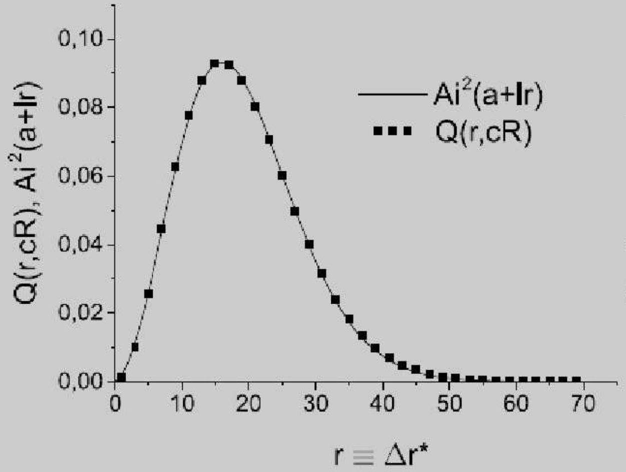
$$\lambda = \frac{\mu}{R}, \qquad \rho = R + r$$

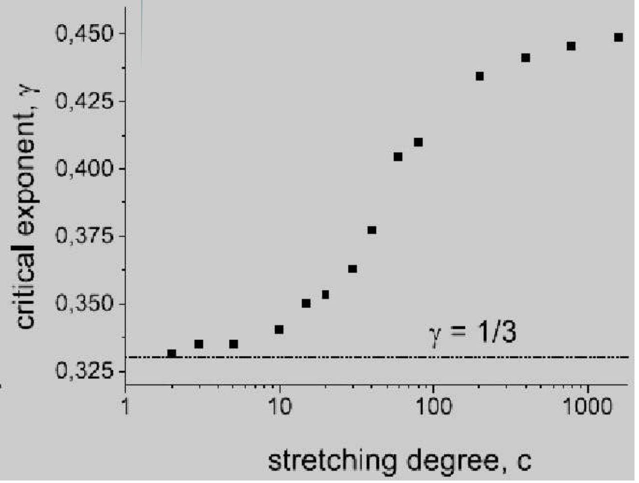
$$Q(r,N) \propto P_N(\rho,\phi|\rho_A,\phi_A) \times P_N(\rho,\phi|\rho_B,\phi_B) = P_N(\rho,\phi|\rho_A,\phi_A)^2$$

$$\mathrm{Var}[r(N)] = \int_0^{\infty} r^2 Q(r, N=cR)\, dr - \left(\int_0^{\infty} r\, Q(r, N=cR)\, dr\right)^2$$

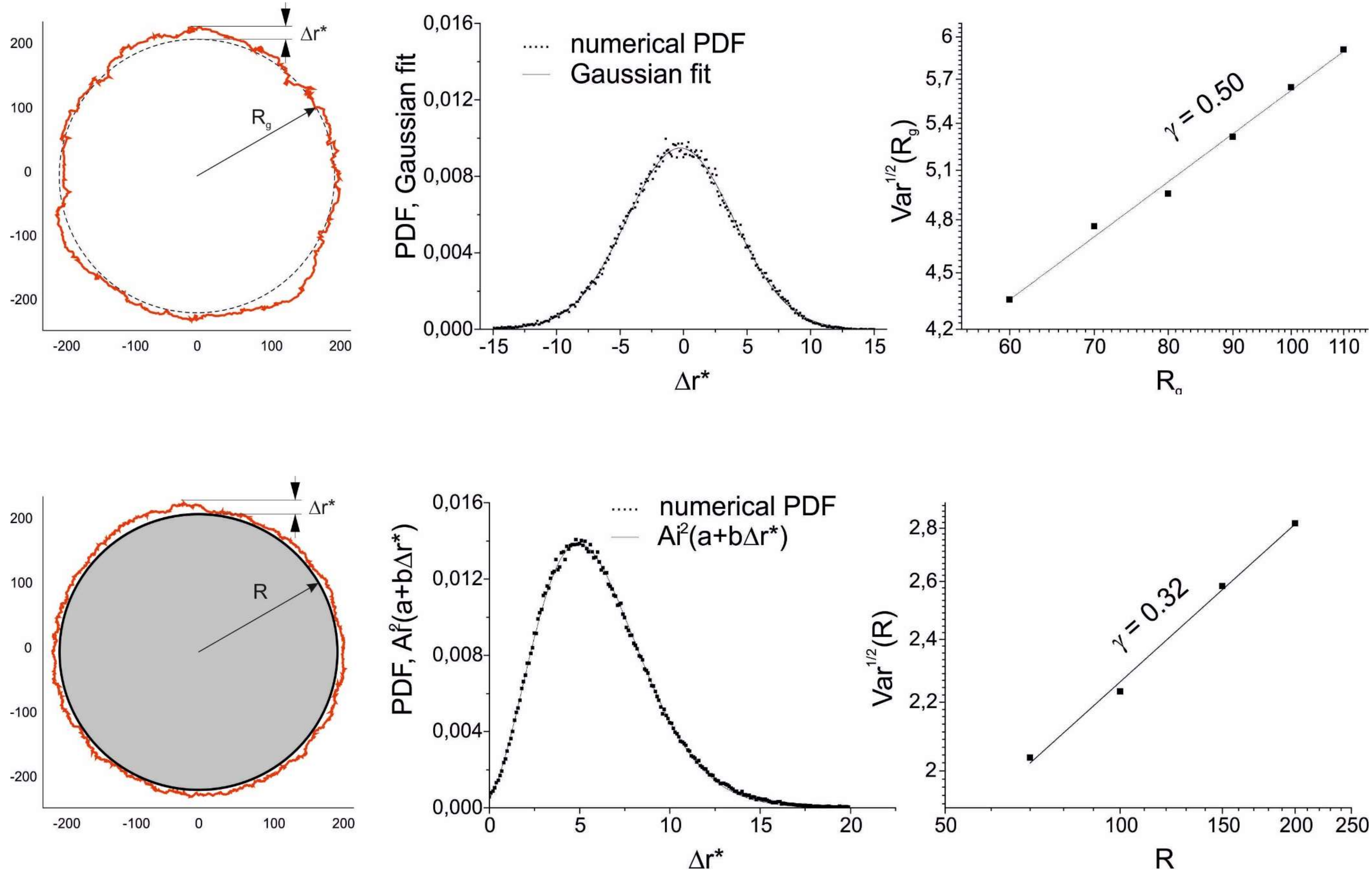
Stretching $N = c\,R$



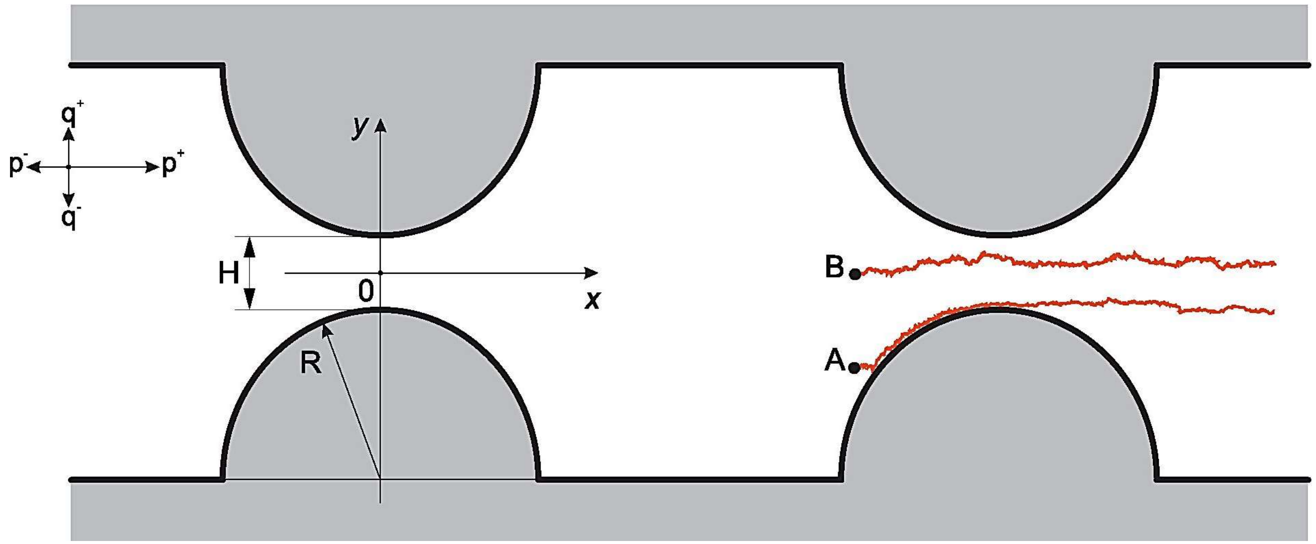


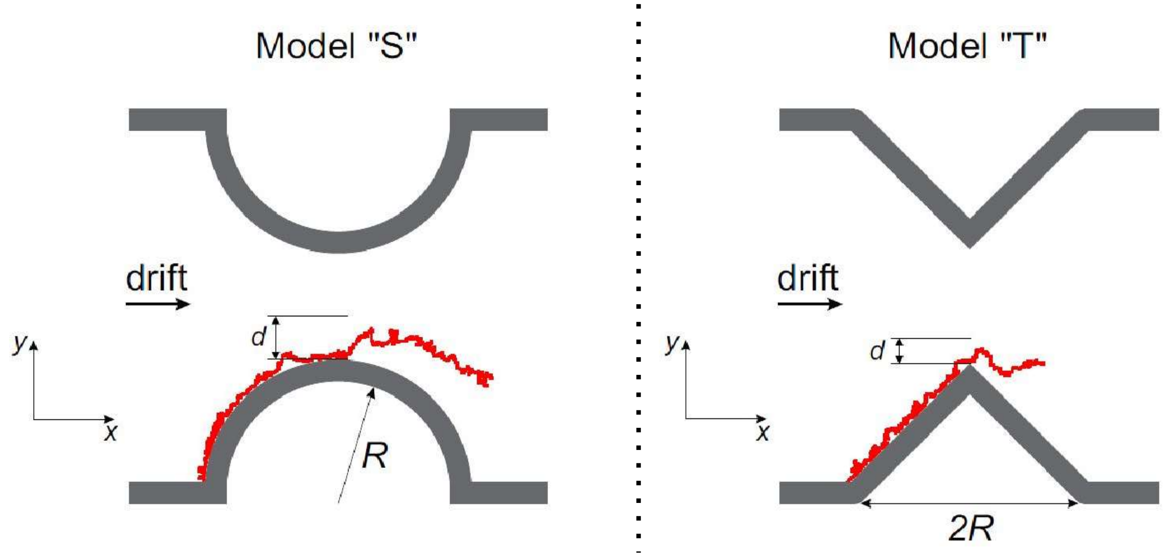


Fluctuations of stretched paths, $\Delta r^*(N) \propto N^{\gamma(c)}$

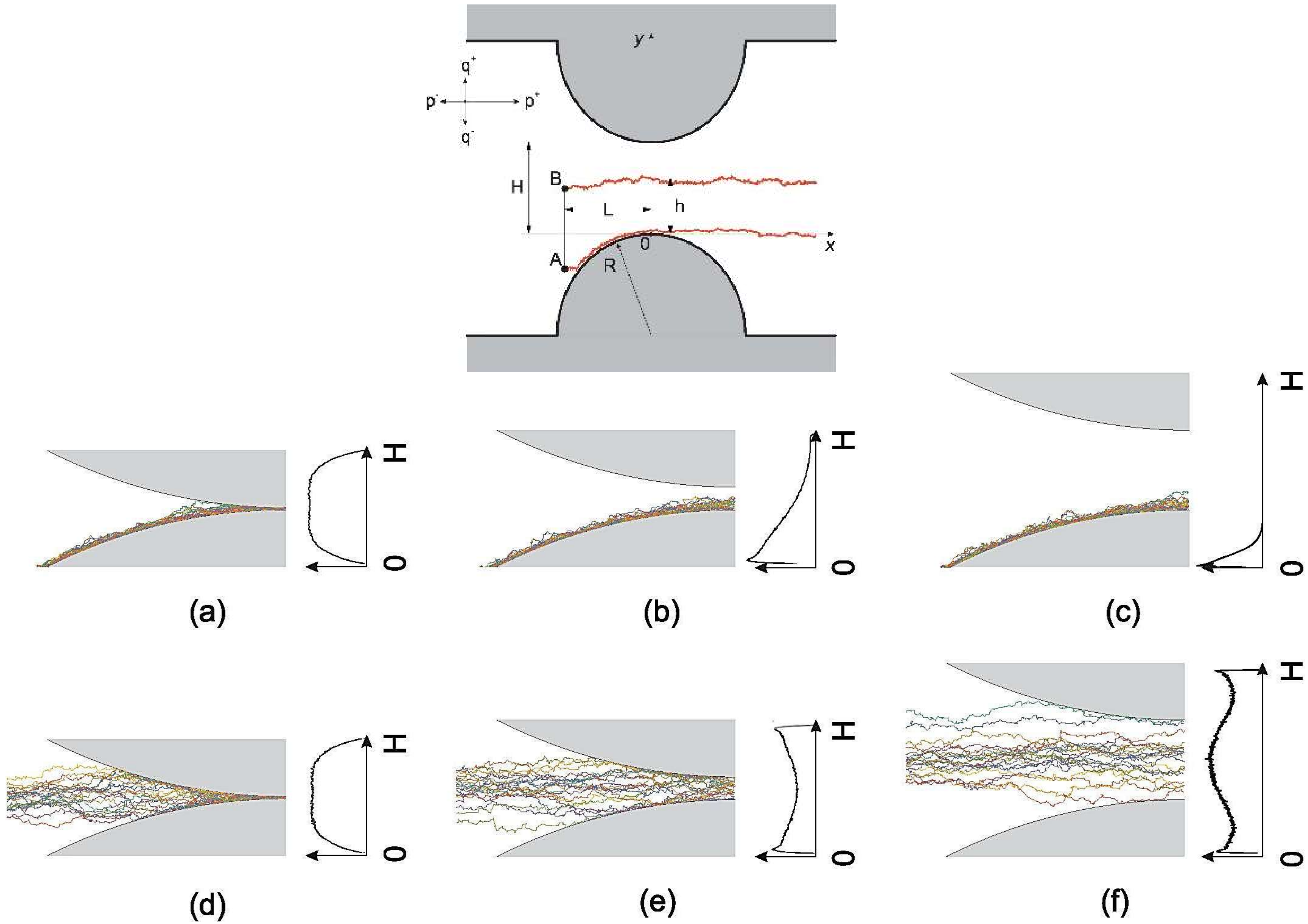


Random walks with longitudinal drift in a two-dimensional non-uniform channel

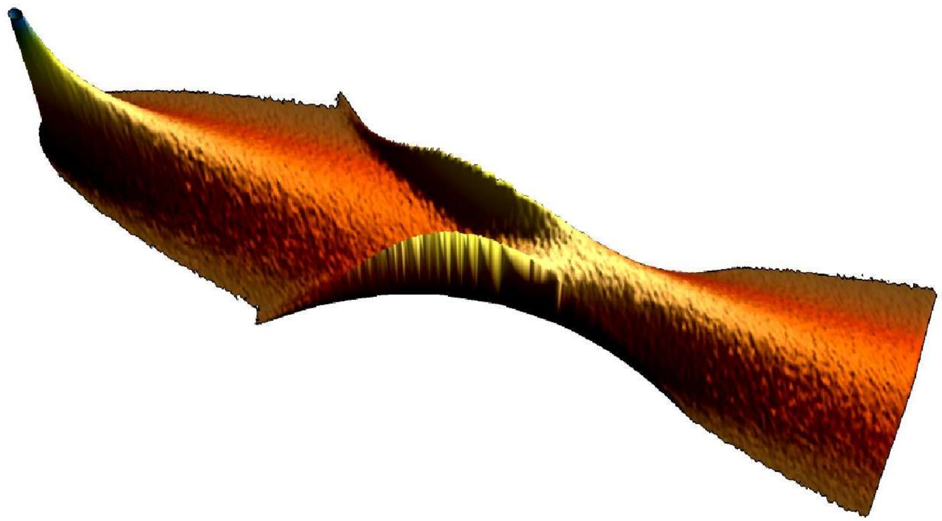


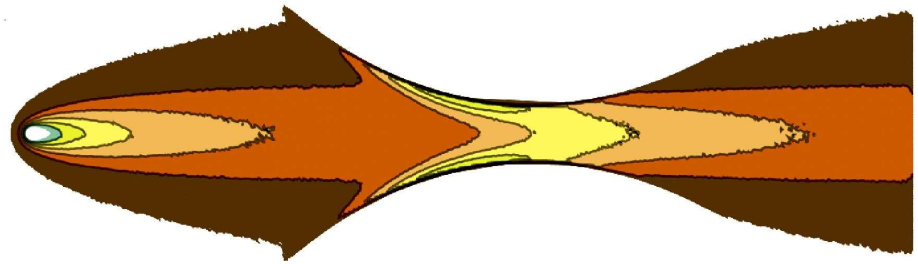


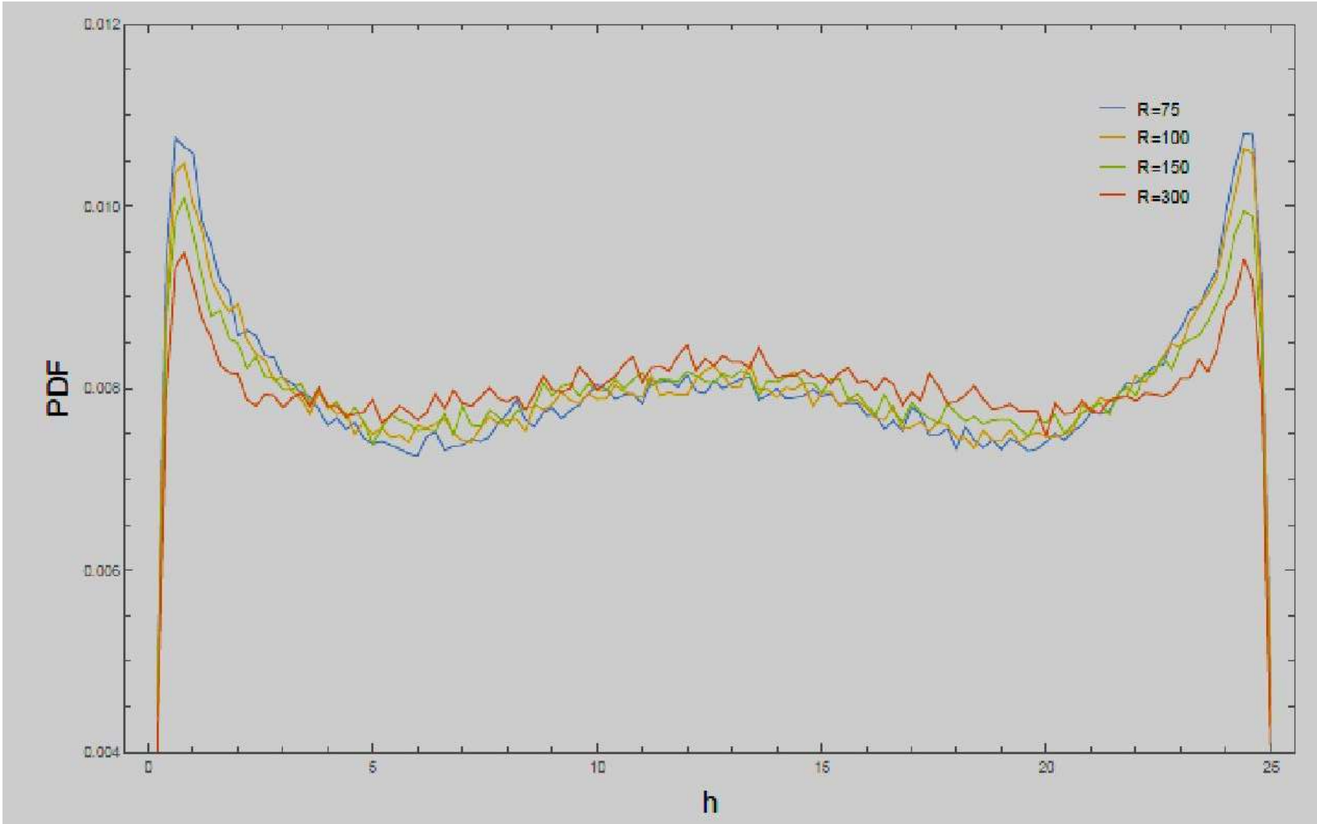
Typical realizations of trajectories of a particle starting away from a funnel and passing through it.



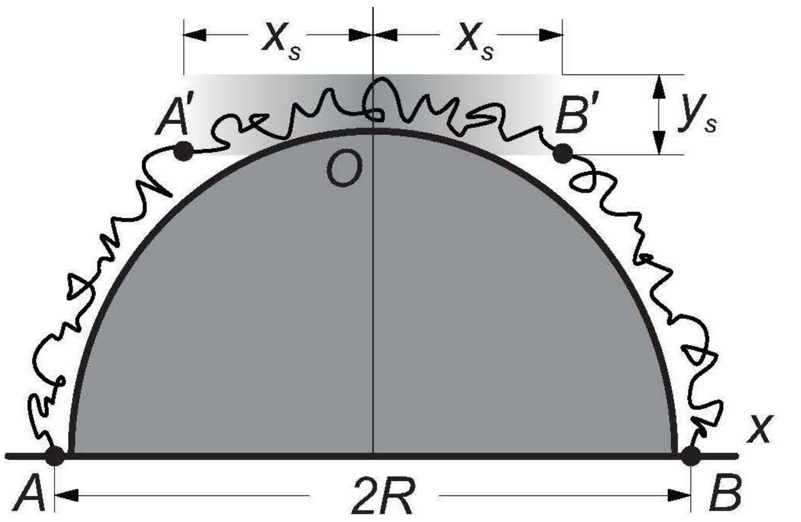
Probability density of paths in a channel's section

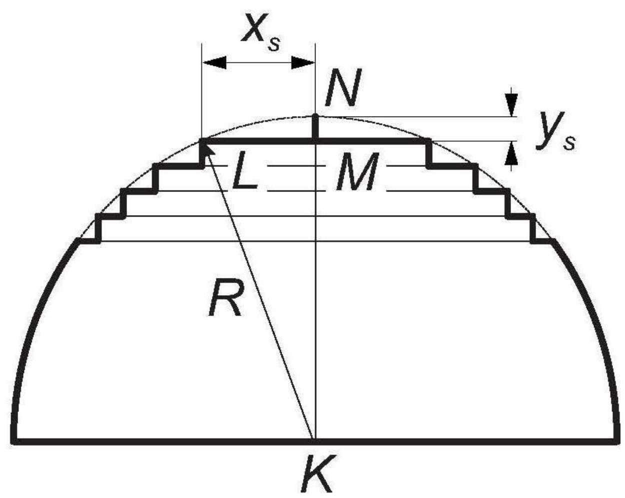






Stretched ($N = c\,R$) paths above the semicircle: scaling consideration





Linearizing curved shape, we get: $x = \sqrt{R^2 - (R-y)^2} \approx \sqrt{2Ry}$

Above the flat line one has a random walk $y \sim \sqrt{x}$

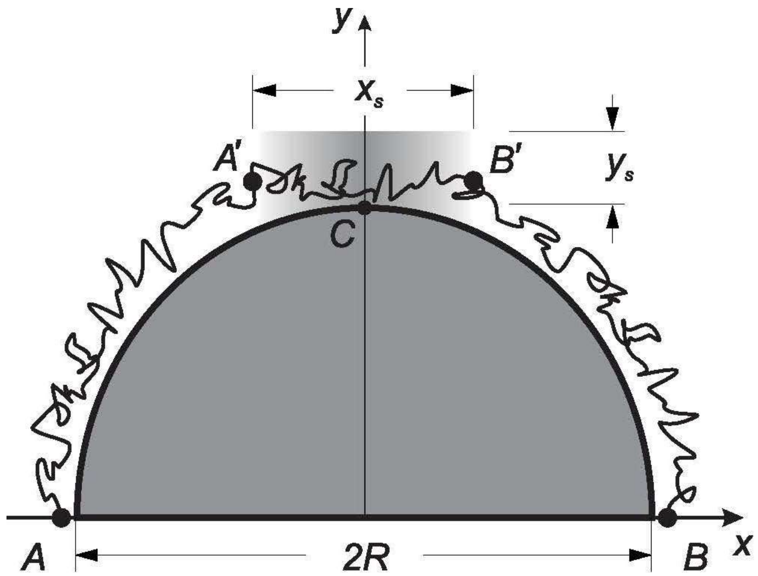
thus $x \sim \sqrt{R\sqrt{x}}$, and finally, $x \sim R^{2/3}, \; y \sim R^{1/3}$

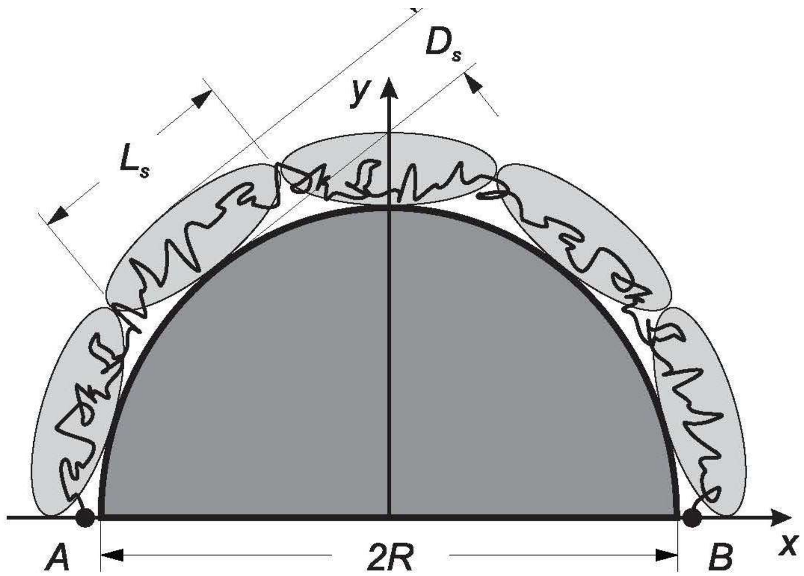
$$\left(\frac{x^2}{R^2} \approx \frac{y}{R} \right)$$

Stretching above algebraic curve $\left(\frac{x}{R} \right)^{\eta} \approx \frac{y}{R}$ provides scaling $y_G(R) \sim R^{\gamma};$

with $\gamma = \frac{\eta - 1}{2\eta - 1}$

Free energy of stretched paths: scaling approach





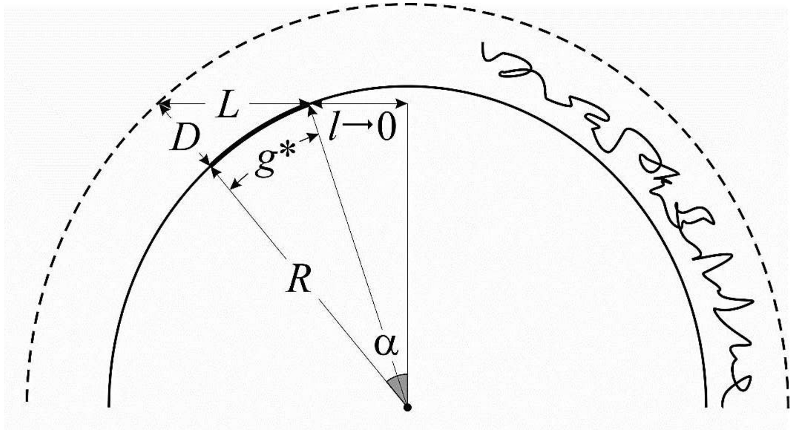
Blob's width is $D_s \sim R^{1/3}$, blob's length is $L_s \sim R^{2/3}$

Free energy of a chain stretched above semicircle $F \sim \dfrac{N}{L_s} \sim \dfrac{R}{R^{2/3}} \sim R^{1/3}$

Gibbs measure can be written as $W(R) = e^{-F(R)} \sim e^{-\alpha R^{1/3}}$

Inverse Laplace transform provides a Lifshitz tail $\dfrac{1}{2\pi i} \oint_C W(R) e^{sR} dR \sim e^{-a^{2/3}/\sqrt{s}}$

Free energy of stretched paths: Optimal fluctuation



Free energy $F(D,R)$ of the stretched ($N = cR$) paths confined in a curved slit of size D, can be estimated as

$$F(D,R) \sim \frac{R}{D^2} - \frac{R}{g^*(R)}$$

where $g^* \sim L$ is the average number of monomers in a blob.

$$\bar{L} = (R + D_s)\sin\alpha - l = (R + D_s)\sin\alpha - \sqrt{R^2 - (R + D_s)^2\cos^2\alpha}$$

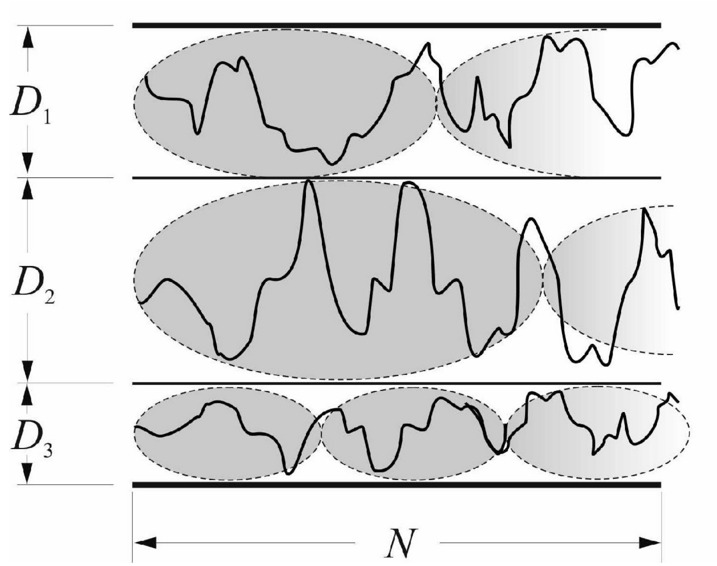
One can estimate g^* as $g^* \sim R\,\alpha_{\min}$, where $\alpha_{\min} \sim \sqrt{\frac{D}{R}}$ which gives the expression for $F(D,R)$:

$$F(D,R) \sim \frac{R}{D^2} - \sqrt{\frac{D}{R}}$$

Minimizing $F(D,R)$ with respect to D, we get equilibrium width of a slit D for stretched paths evading the semicircle: $\bar{D}(N) \sim N^{1/3}$

2. Statistics of one-dimensional random walks in a Poissonian field of traps

Lifshitz tail in optimal fluctuation for survival probability in a one-dimensional Poissonian field of random segments



Estimate survival probability in an ensemble of random intervals D with the distribution $Q(D) \sim p^D$

Non-equilibrium free energy to be minimized over D:

$$F(N, D) = F_e(N) - \ln Q(D)$$

where $F_e(D,N) \sim \dfrac{N}{D^2}; \quad \ln Q(D) \sim D \ln p \quad (\ln p = \beta)$

Correspondingly, $\bar{D}(N) \sim N^{1/3}$ and survival probability (Balagurov, Vaks, 1974) is

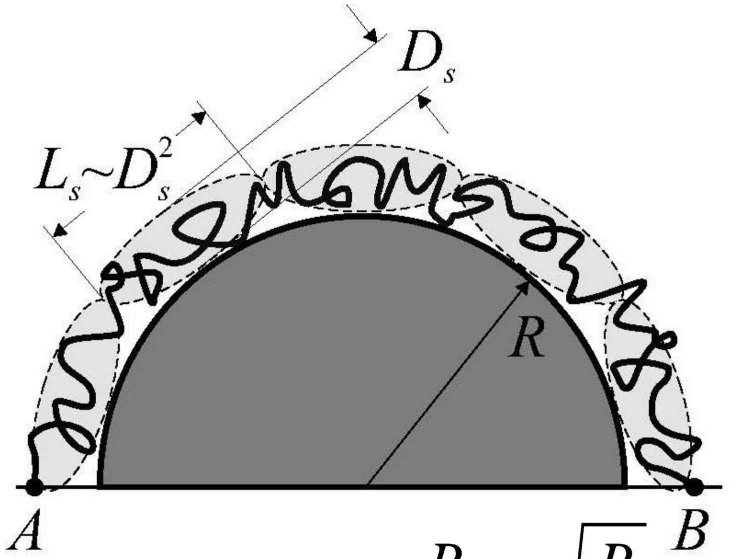
$$P(N) = e^{-F(R,\bar{D})} \sim e^{-\beta^{2/3} N^{1/3}}$$

The inverse Laplace transform gives the Lifshitz tail of 1D Anderson localization

$$P(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} P(N) e^{sN} \sim e^{-\beta/\sqrt{s}}$$

Comparison of disorder-free stretched KPZ exponent above convex boundary and Lifshitz tail in a Poissonian field

Gibbs measure of stretched path in curved channel



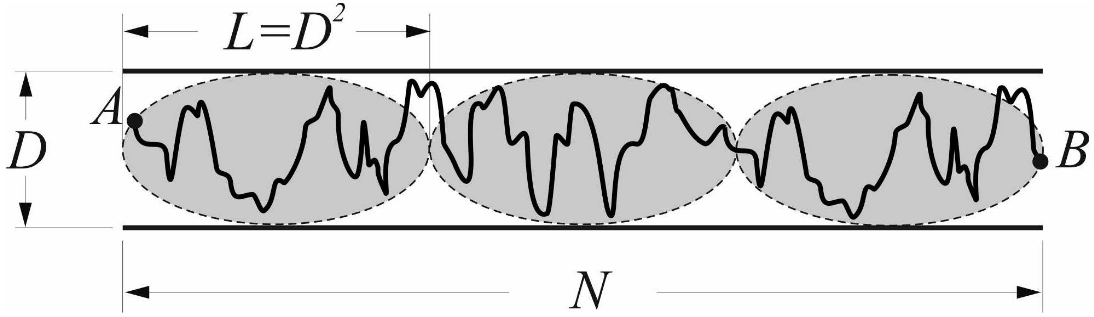
$$F(D,R) \sim \frac{R}{D^2} - \sqrt{\frac{R}{D}}$$

Tube width is $\bar{D}(N) \sim N^{1/3}$

No disorder, however path is stretched above semicircle

$$W(R \sim N) \sim e^{-\alpha N^{1/3}}$$

Survival probability in 1D trapping in Poissonian field



Free energy has to be minimized over D

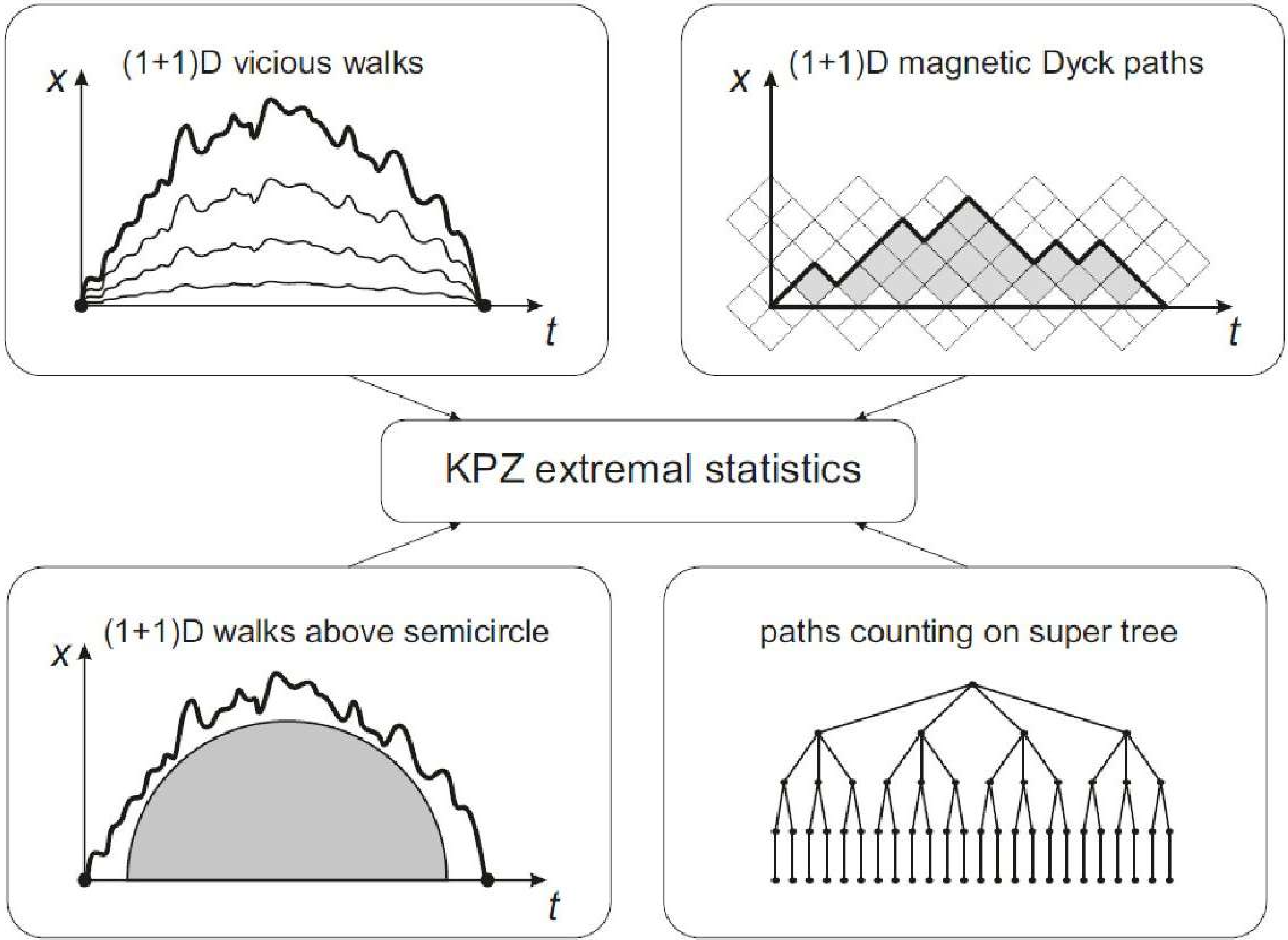
$$F(N,D) = F_e(N) - \ln Q(D) \sim \frac{N}{D^2} - \beta D$$

In Poissonian disorder one gets

$$\bar{D}(N) \sim N^{1/3}$$

The survival probability is

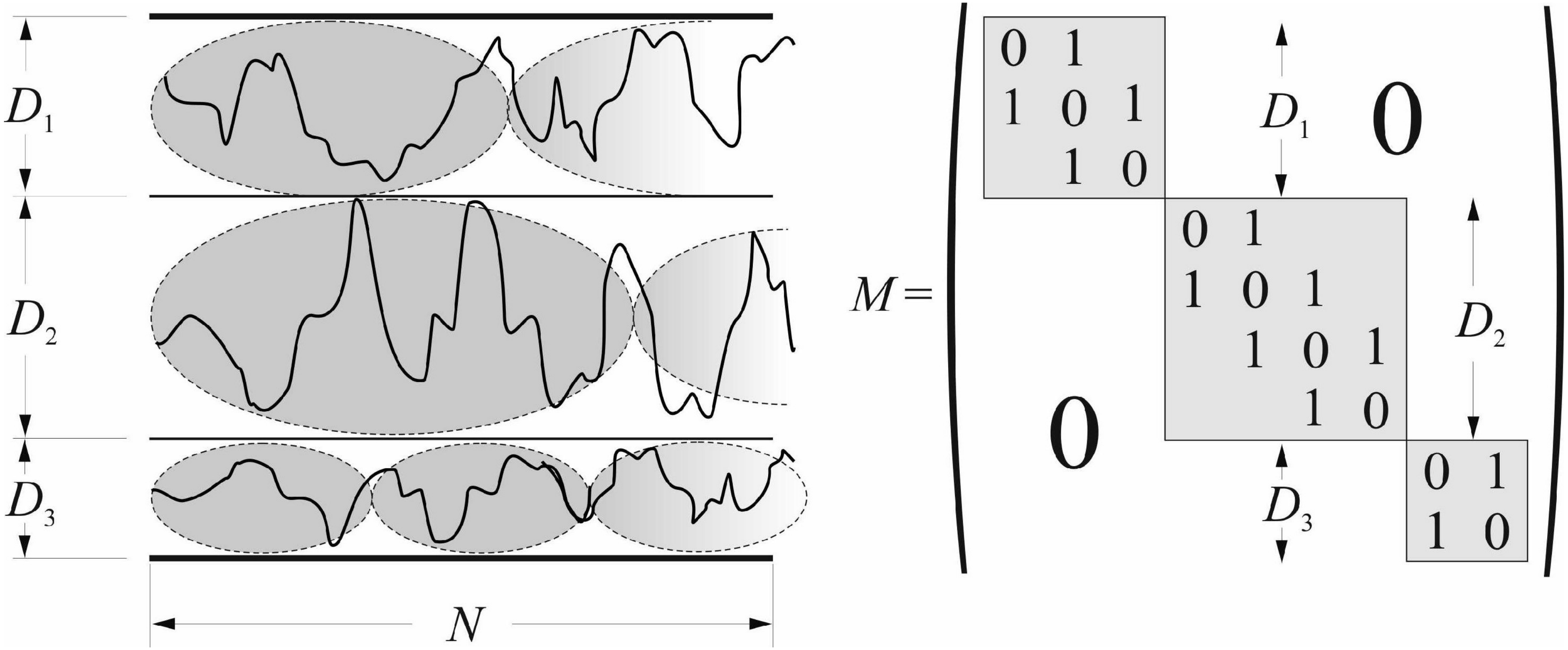
$$P(N) = e^{-F(R,\bar{D})} \sim e^{-\beta^{2/3} N^{1/3}}$$



(1+1)D vicious walks
x
t
(1+1)D magnetic Dyck paths
x
t
KPZ extremal statistics
(1+1)D walks above semicircle
x
t
paths counting on super tree

3. Spectral properties of random operators

Adjacency matrix splits into cells with the distribution $Q(D) \sim p^D \; (0 < p < 1)$



Matrix N x N ($N \gg 1$)

Set of eigenvalues in the cell of size D is

$$\lambda_k(D) = -2\cos\frac{\pi k}{D+1} \qquad (k = 1, \ldots, D)$$

Spectral statistics of Schrödinger-like random operators

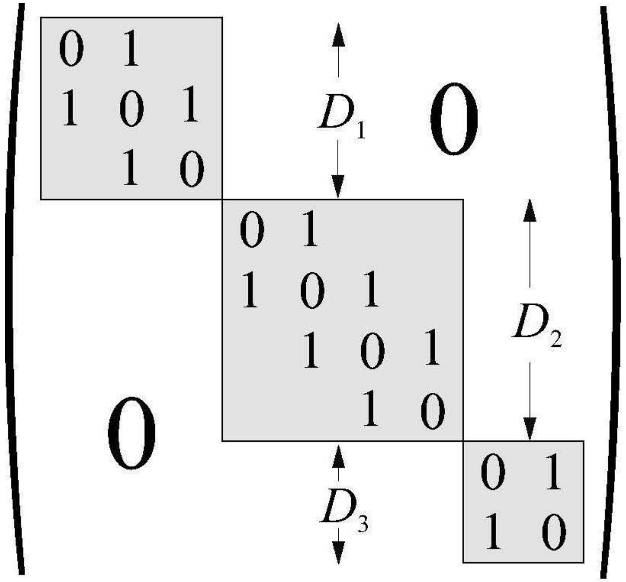
Consider an ensemble of two (three)-diagonal matrices

$$A_N = \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ x_1 & 0 & x_2 & & \\ 0 & x_2 & 0 & & \\ \vdots & & & & \\ & & & & x_{N-1} \\ 0 & & & x_{N-1} & 0 \end{pmatrix}$$

where the matrix elements are:

$$x_k = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1 - p \end{cases}$$

Spectral density of adjacency matrix



Spectral density, $\rho(\lambda)$, of ensemble of random matrices is:

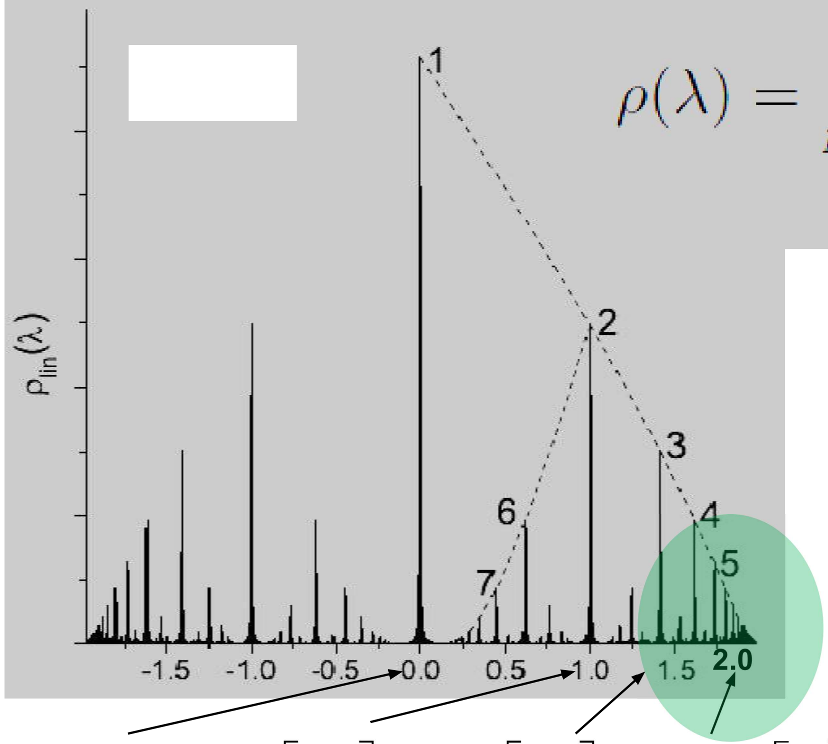
$$\rho(\lambda) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_{n=1}^{N} \sum_{k=1}^{D} \delta(\lambda - \lambda_{k,D}) \right\rangle$$

$$= \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{\varepsilon}{\pi N} \sum_{n=1}^{N} Q_D \sum_{k=1}^{D} \mathrm{Im} \frac{1}{\lambda - \lambda_{k,D} - i\varepsilon}$$

Counting contributions from exponentially weighted cells, we get:

$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{1}{\pi N} \sum_{n=1}^{N} p^D \sum_{k=1}^{D} \frac{\varepsilon}{\left(\lambda - 2\cos\frac{\pi k}{D+1}\right)^2 + \varepsilon^2}$$

Lifshitz tail of the spectral density $\rho(\lambda)$ at the **spectral edge**



$-2\cos\left[\frac{\pi}{2}\right]$ $-2\cos\left[\frac{2\pi}{3}\right]$ $-2\cos\left[\frac{3\pi}{4}\right]$ ... $-2\cos\left[\frac{\pi}{1}\right]$

$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{1}{\pi N} \sum_{n=1}^{N} p^n \sum_{k=1}^{n} \frac{\varepsilon}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$

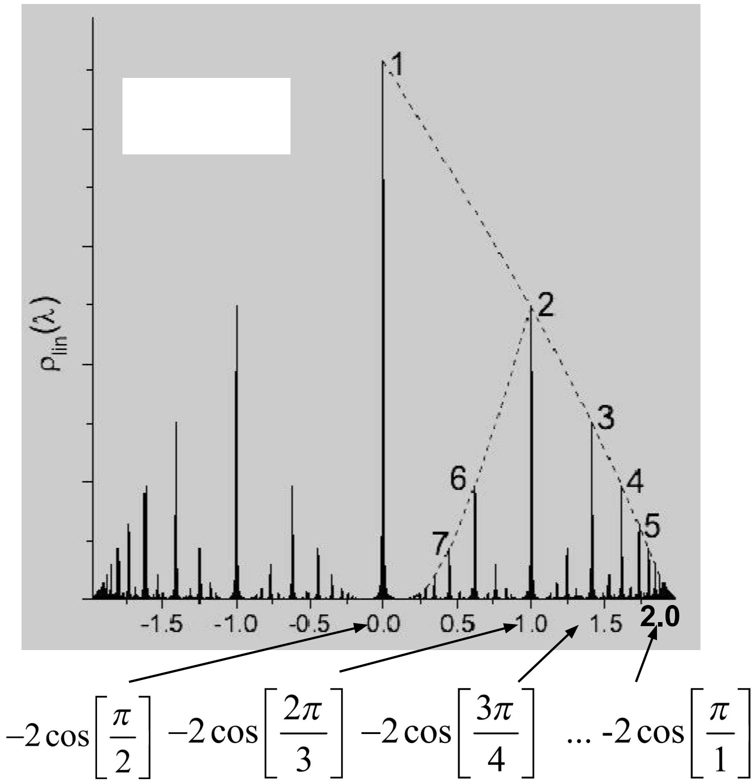
$$\lambda = -2\cos\frac{\pi D}{D+1}\Big|_{D\to\infty} \to 2 - \frac{\pi^2}{D^2},$$

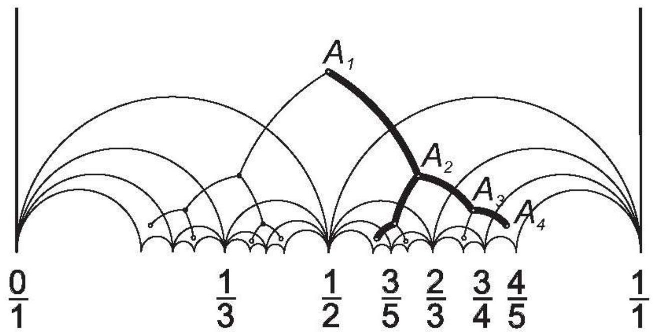
$$\rho^{S_1}(\lambda_k)\Big|_{k\to\infty} \to p_k^D$$

Lifshitz tail of 1D Anderson localization

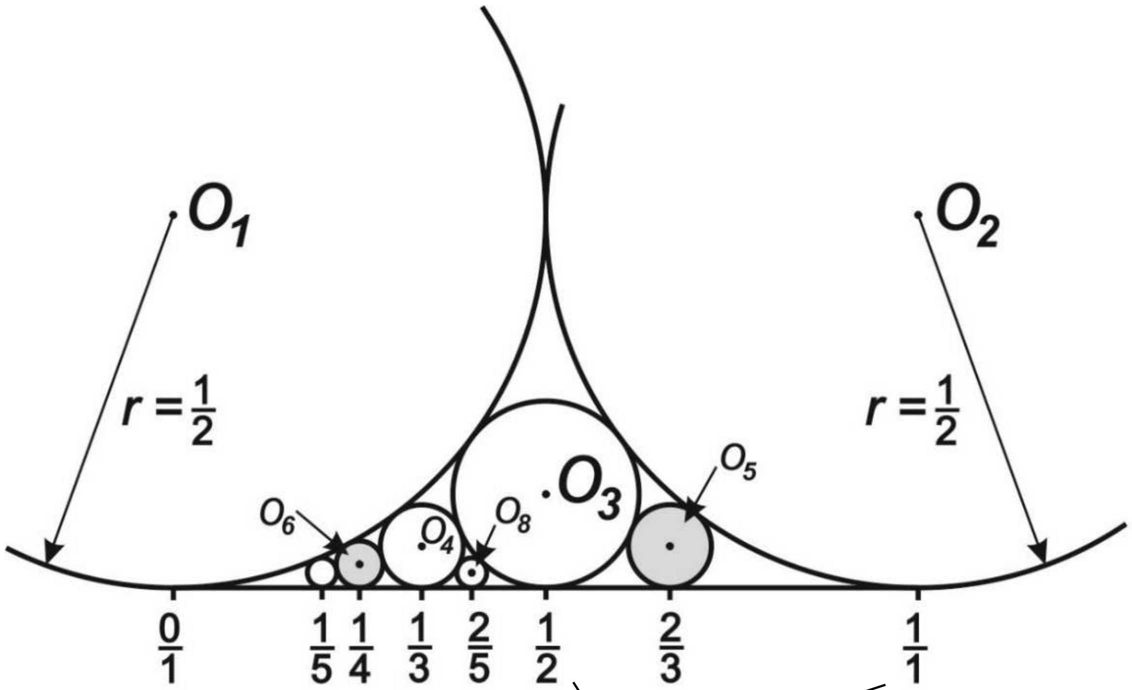
$$\rho(\lambda \to 2) \to p^{\pi/\sqrt{2-\lambda}}$$

Positions of peaks are defined by composition rules for Farey numbers (Ford circles)





(b)



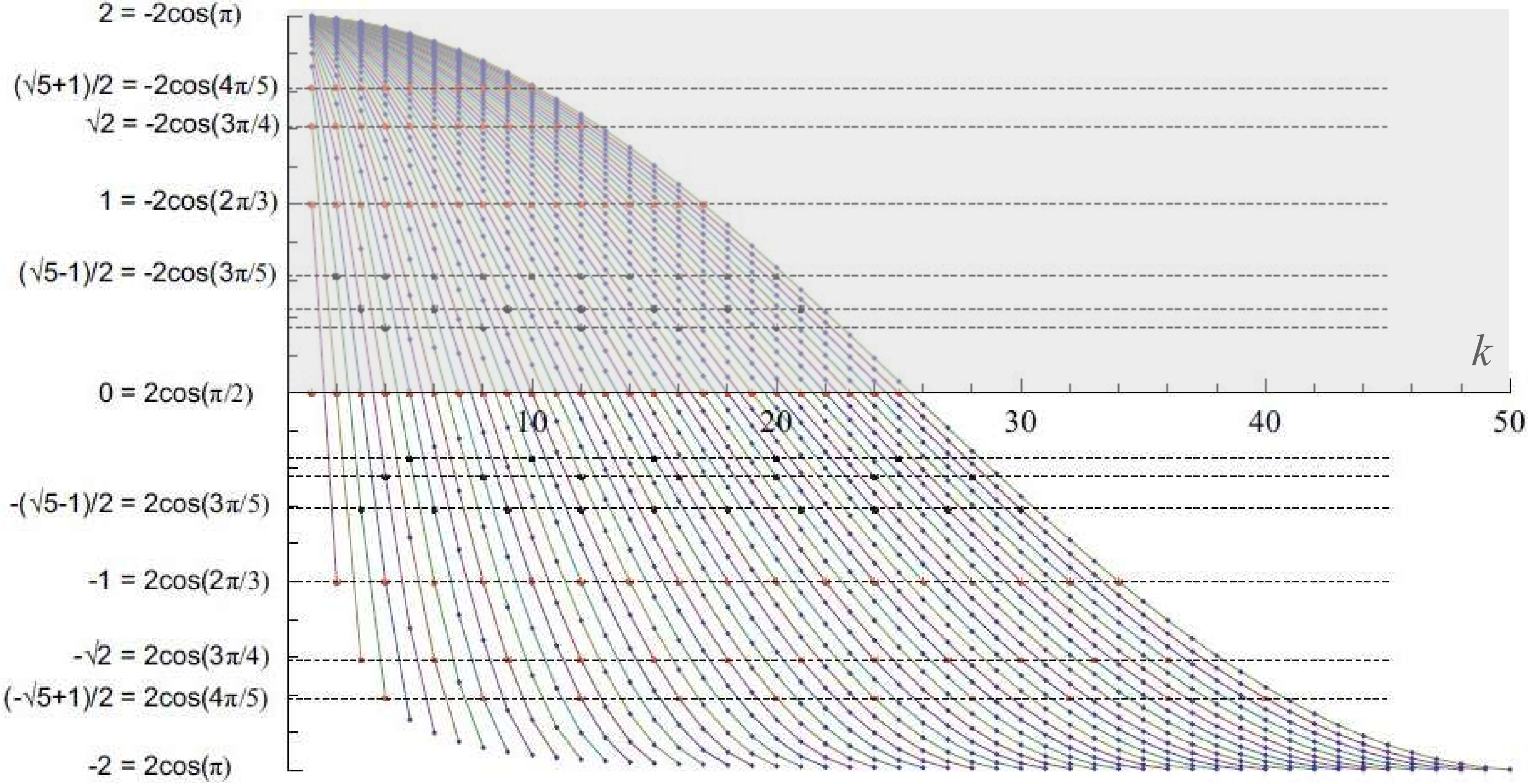
$$\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$$

$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} \oplus \frac{p_{n+1}}{q_{n+1}} = \frac{p_{n-1} + p_{n+1}}{q_{n-1} + q_{n+1}}$$

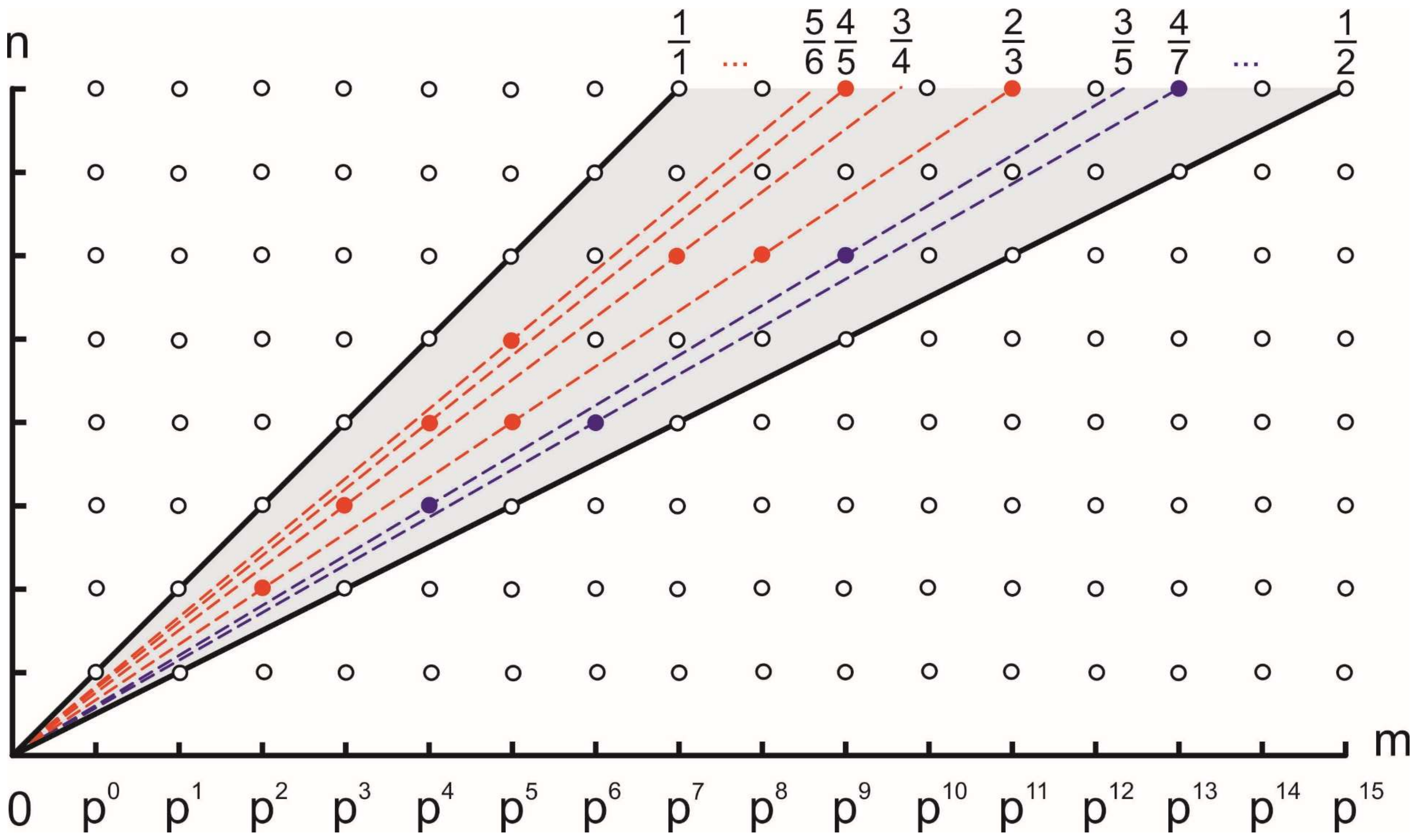
How to compute the amplitude of a peak (degeneracies of eigenvalues)

$$\lambda_k(n) = -2\cos\frac{\pi k}{n+1}$$

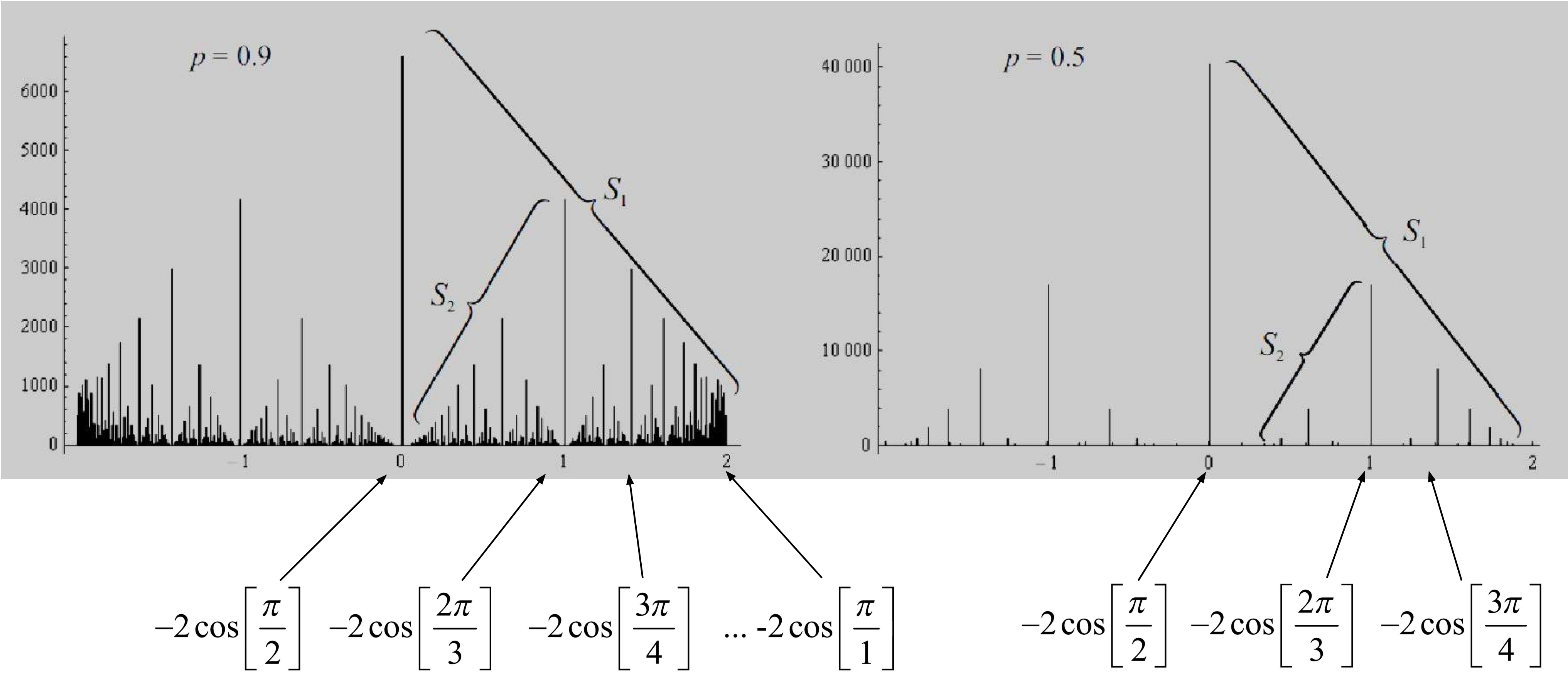
$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{1}{\pi N} \sum_{n=1}^{N} p^n \sum_{k=1}^{n} \frac{\varepsilon}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$



How to compute the amplitude of a peak (degeneracies of eigenvalues)



Sample spectral densities for $p = 0.9$ and $p = 0.5$

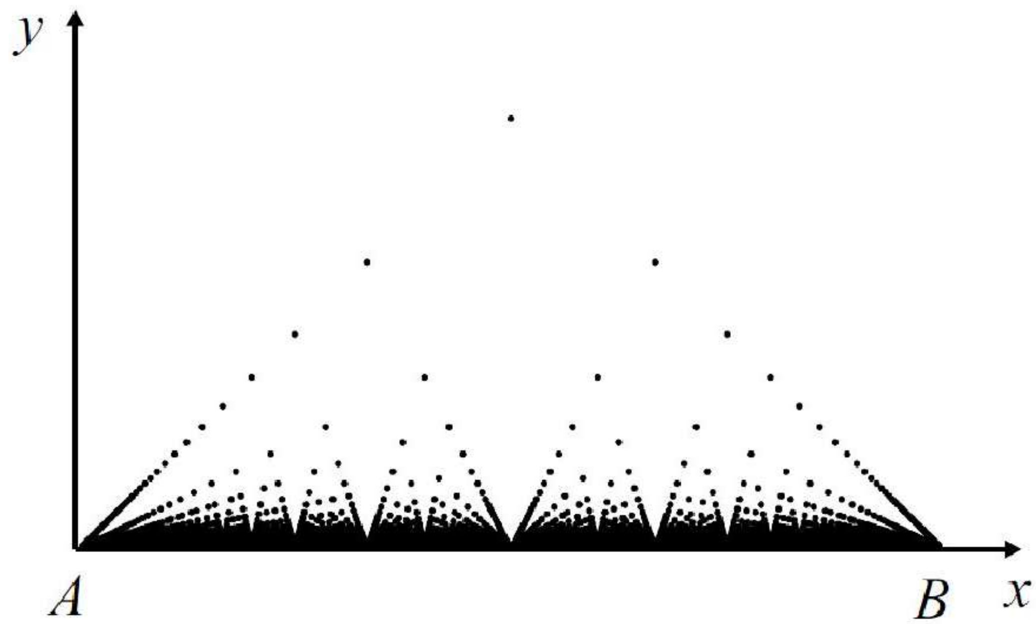


$$S_1 : \begin{cases} \lambda_1 = -2\cos\frac{\pi}{2} & \rho_{S_1}(\lambda_1) = p^1 + p^3 + p^5 + p^9 + \dots \\ \lambda_2 = -2\cos\frac{2\pi}{3} & \rho_{S_1}(\lambda_2) = p^2 + p^5 + p^8 + p^{11} + \dots \\ & \dots \\ \lambda_n = -2\cos\frac{\pi n}{n+1} & \rho_{S_1}(\lambda_n) = \sum_{s=1}^{\infty} p^{(n+1)s-1} = \frac{p^n}{1-p^{n+1}} \quad (k = 1, 2, \dots) \end{cases}$$

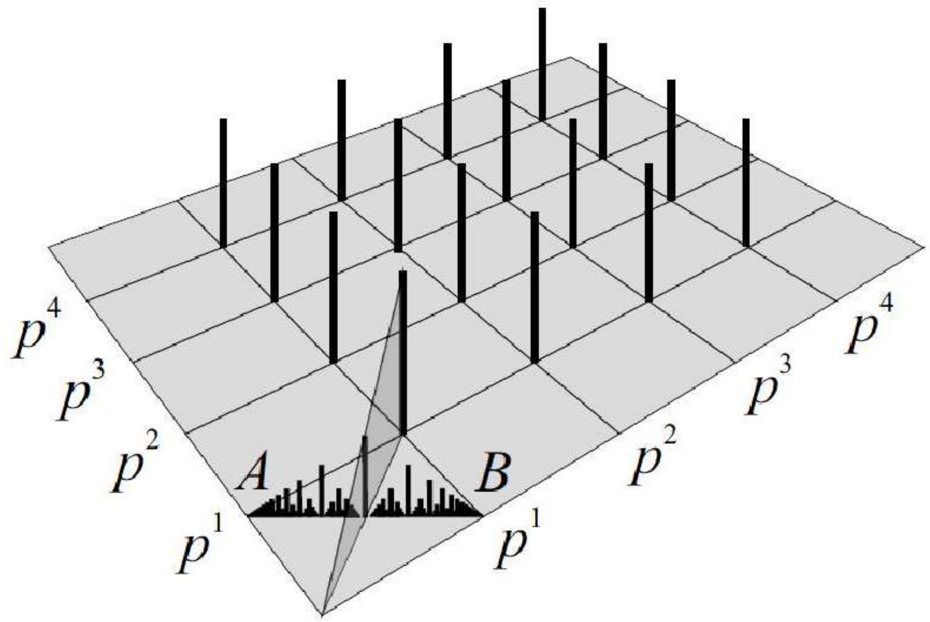
Discontinuous Riemann "raindrop" function and its regularization

$$g(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n}, \text{ and } (m,n) \text{ coprime} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

View of the Riemann function

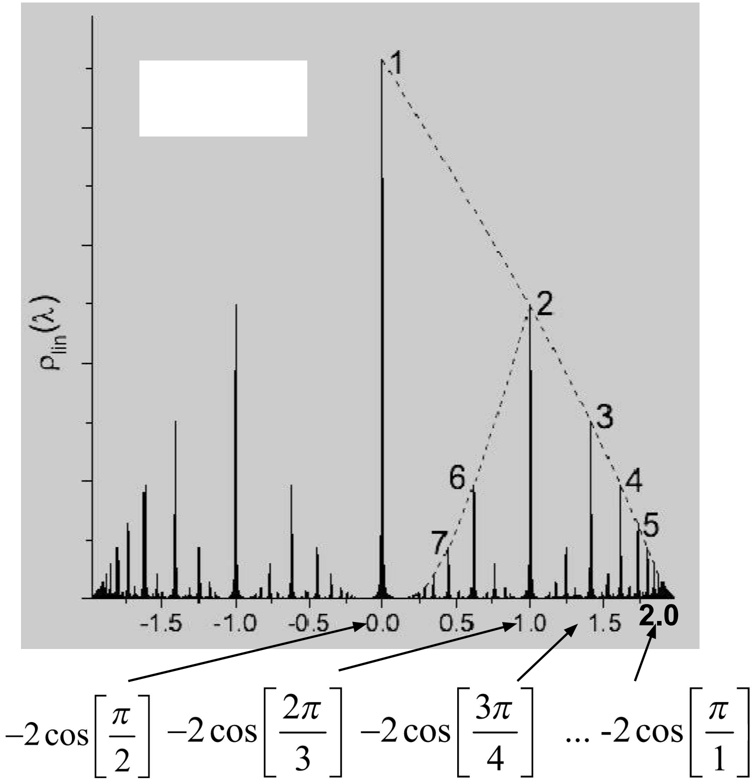


Riemann function as an "Euclid Orchard"



Expression for the spectral density $\rho(\lambda)$

$$\rho(\lambda) = \frac{p^{1/g(u)}}{1 - p^{1+1/g(u)}}; \qquad u = \frac{1}{\pi} \arccos \frac{\lambda}{2}$$



Spectral tail for $p < 1$

$$S_1 : \lambda = -2\cos\frac{\pi k}{k+1}\Big|_{k\to\infty} \to 2 - \frac{\pi^2}{k^2},$$

$$\rho^{S_1}(\lambda_k)\Big|_{k\to\infty} \to p^k$$

Lifshitz tail of 1D Anderson localization

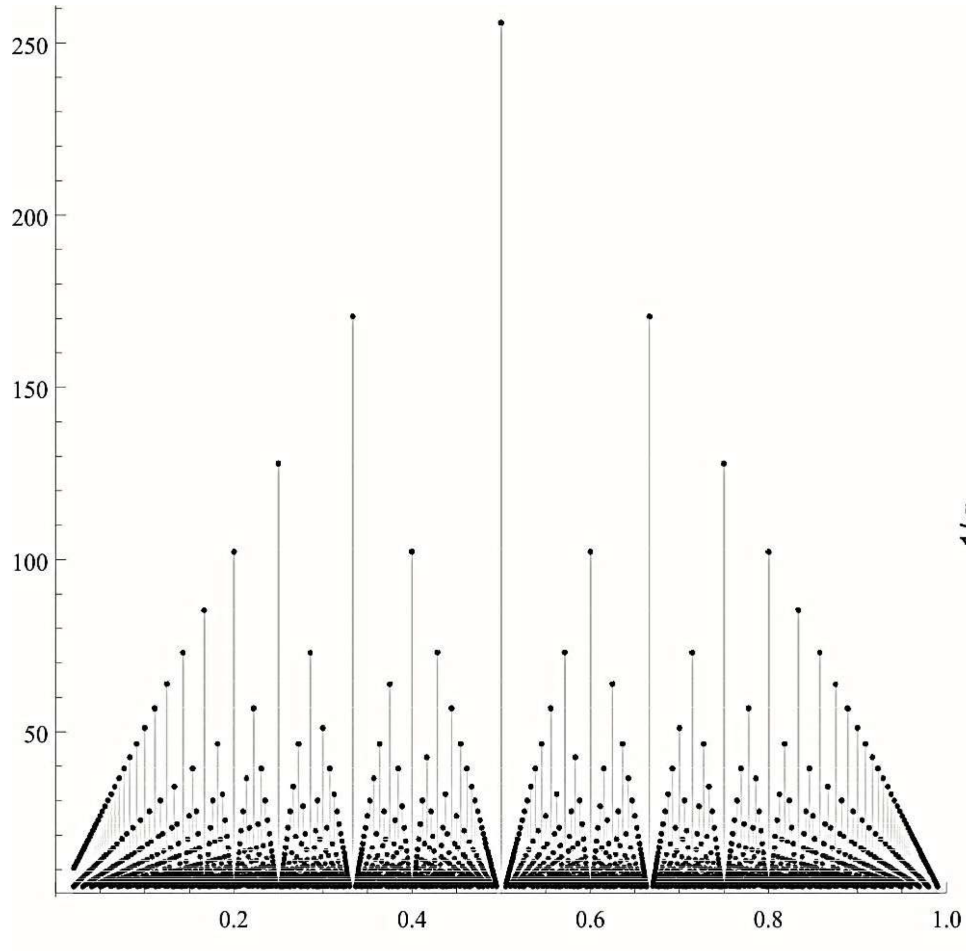
$$\rho(\lambda \to 2) \to p^{\pi/\sqrt{2-\lambda}}$$

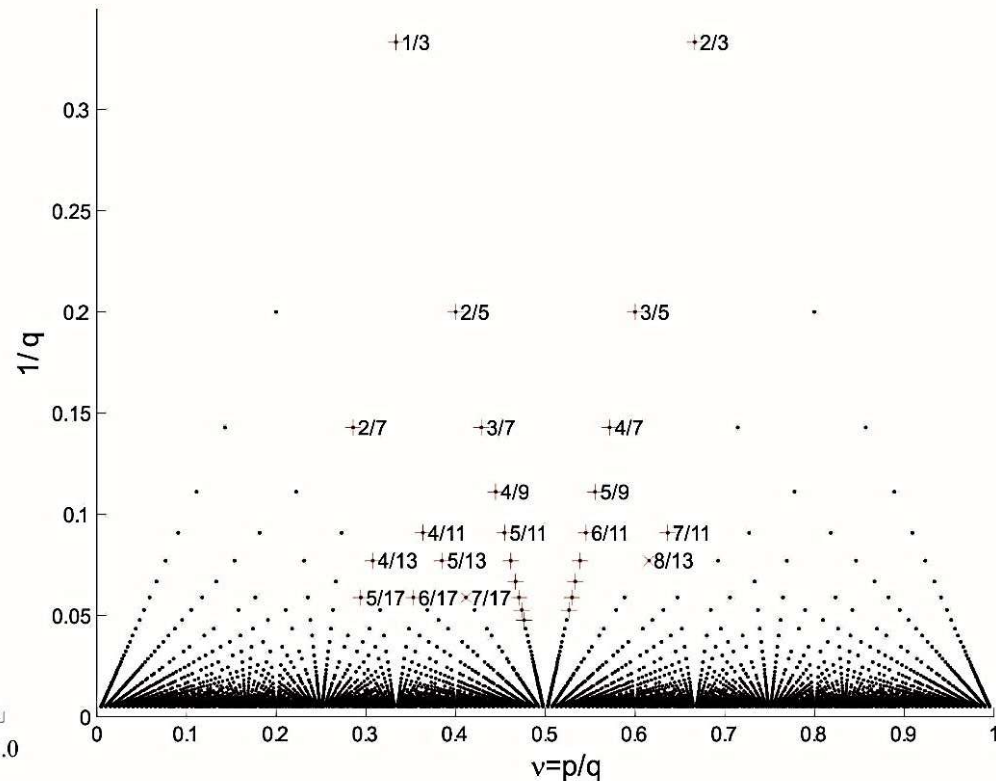
Laplace transform gives:

$$\rho(N) = \int \rho(\lambda) e^{-\lambda N} d\lambda\Big|_{N>>1} \sim \varphi(N) e^{-aN - bN^{1/3}}$$

Regularization of a normalized Riemann "raindrop" function by

$$f_1(x) = \left(\frac{\pi}{12\varepsilon}\right)^{1/2} \sqrt{-\ln|\eta(x+iy)|}$$





The "stability" diagram of Tao-Thouless Fractional Quantum Hall states [from E.J. Bergholtz et al, 2008]: the lower the disorder, the more fractions are observed

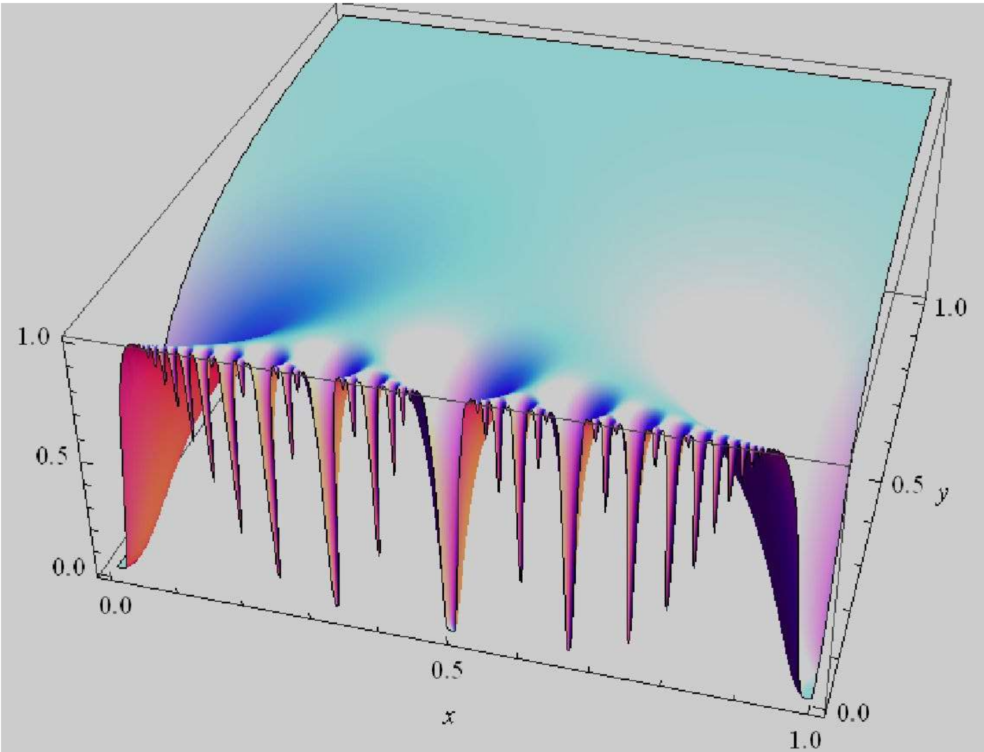
Reminder: Dedekind η – function

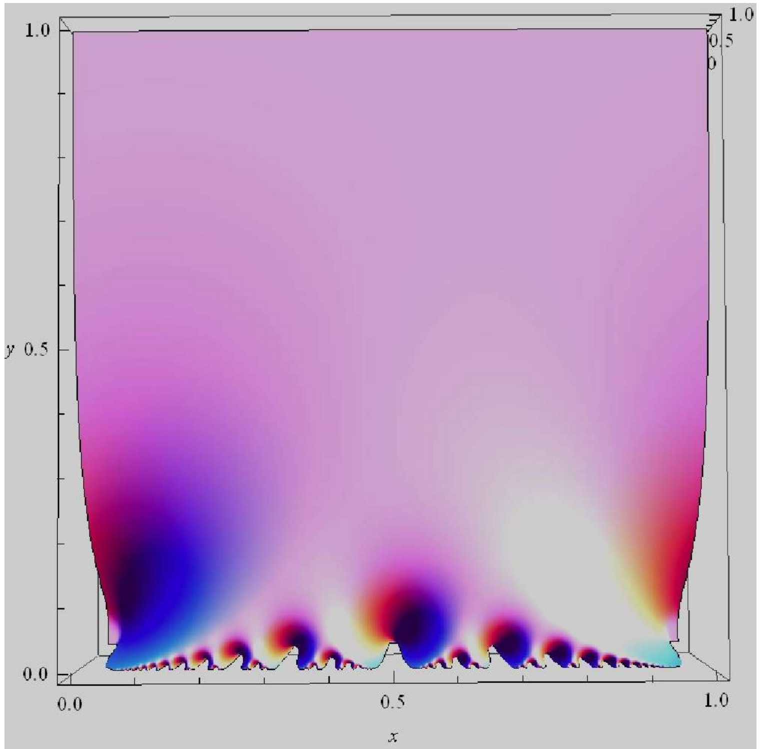
$$\eta(z) = e^{\pi i z/12} \prod_{n=0}^{\infty} (1 - e^{2\pi i n z}); \quad z = x + iy \quad (y > 0)$$

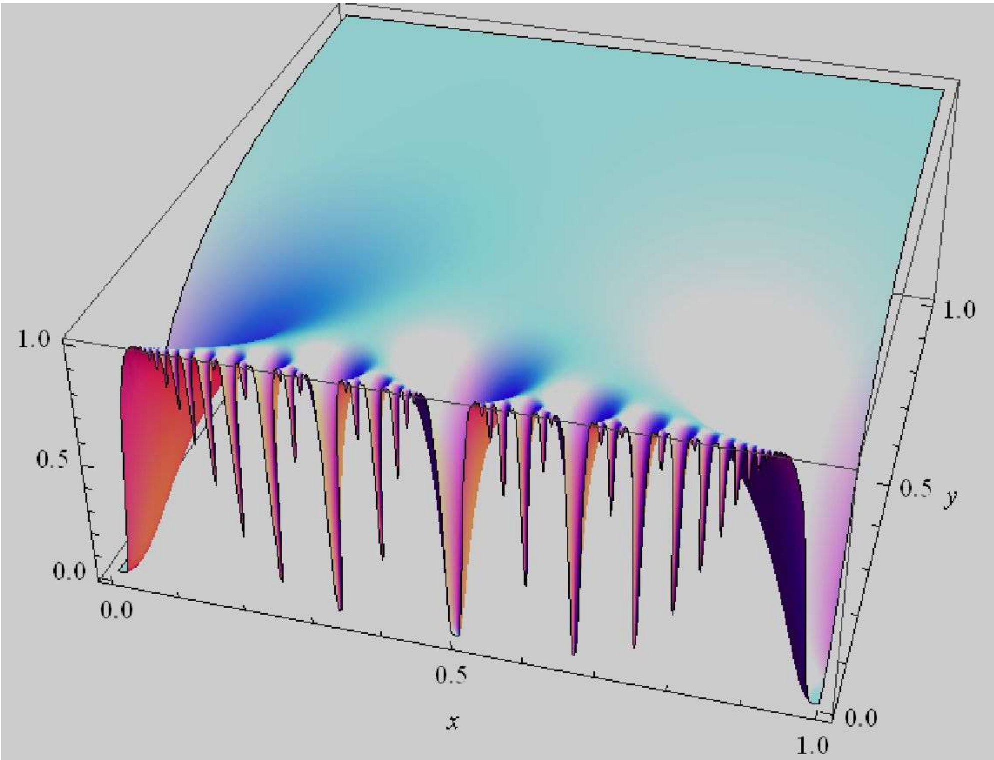
$$\eta(z+1) = e^{\pi i z/12}\, \eta(z)$$

$$\eta\left(-\tfrac{1}{z}\right) = \sqrt{-i}\, \eta(z)$$

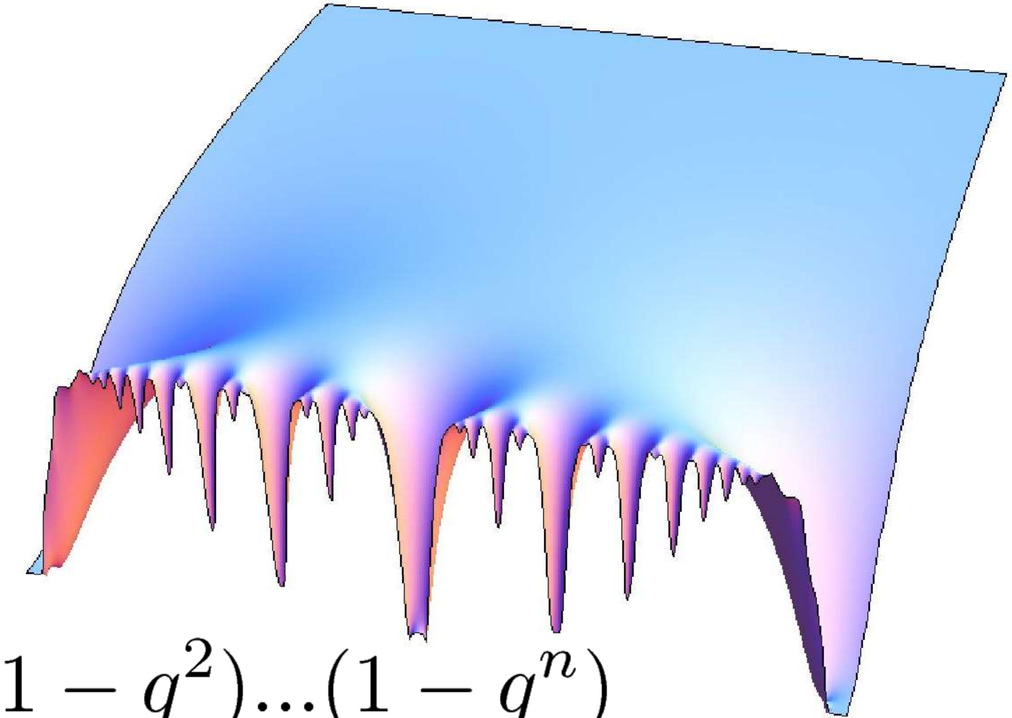
Define $f(z) = \text{const}\,|\,\eta(x+iy)\,|\, y^{1/4}$







Dedekind η -function



Zagier-Kontsevich “strange” functior

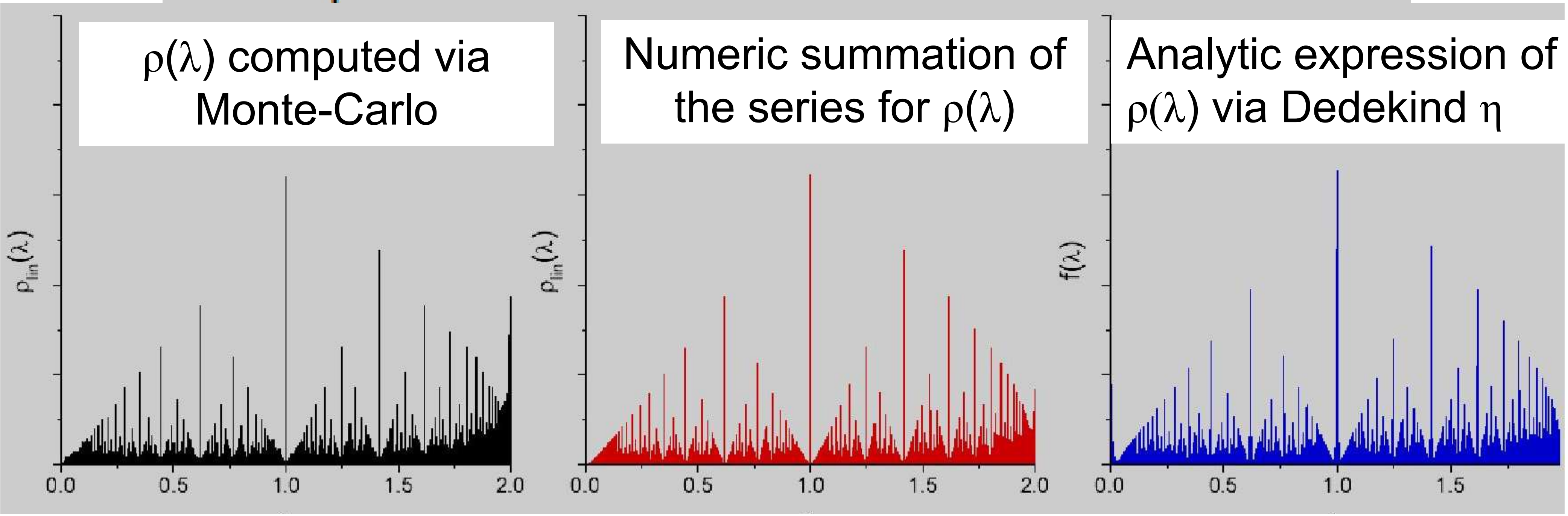
$$F(q) = \sum_{n=1}^{\infty}(1-q)(1-q^2)...(1-q^n)$$

D. Zagier, Vasilyev Invariants and strange identity related to Dedekind eta-function, Topology (2001)

Spectral density $\rho(\lambda)$ of ensemble of exponentially weighted random 3-diagonal matrices at $p \to 1$

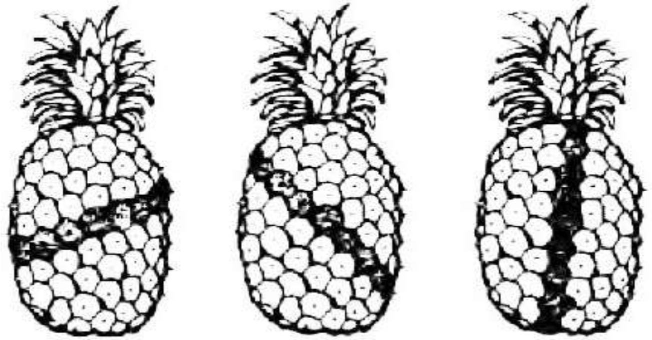
$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{\varepsilon}{\pi N} \sum_{n=1}^{N} p^n \sum_{k=1}^{n} \frac{1}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$

$$\lim_{p\to 1^-} \frac{\rho(\lambda, p)}{\sqrt{-\ln\left|\eta\left(\frac{1}{\pi}\arccos(-\lambda/2) + i\frac{(1-p)^2}{12\pi}\right)\right|}} = 1$$



4. Connection with phyllotaxis

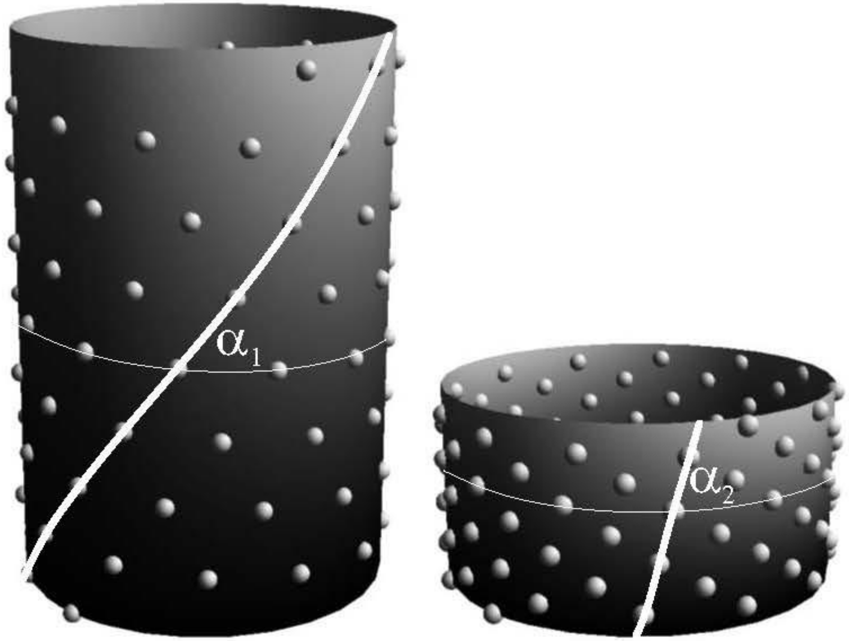
Phyllotaxis

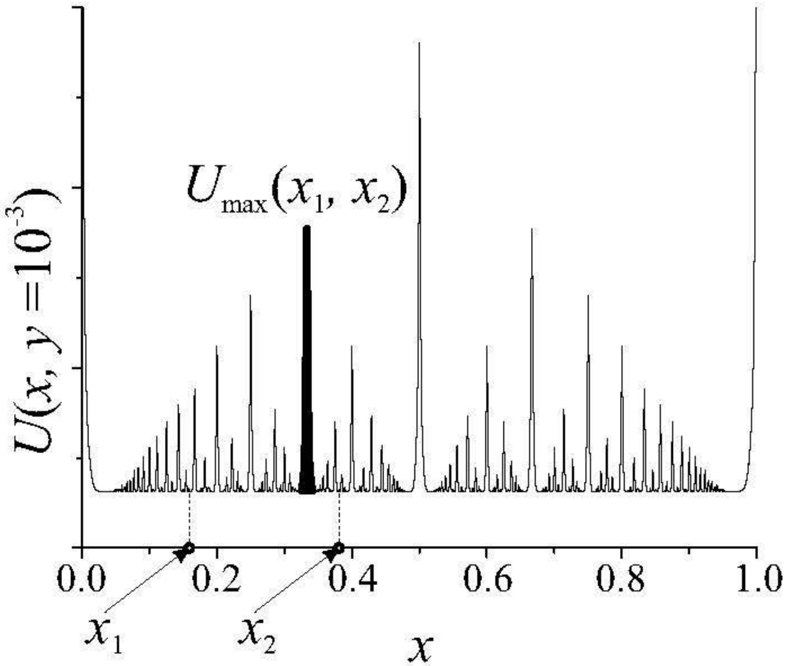


Energetic approach to phyllotaxis, L. Levitov, 1991

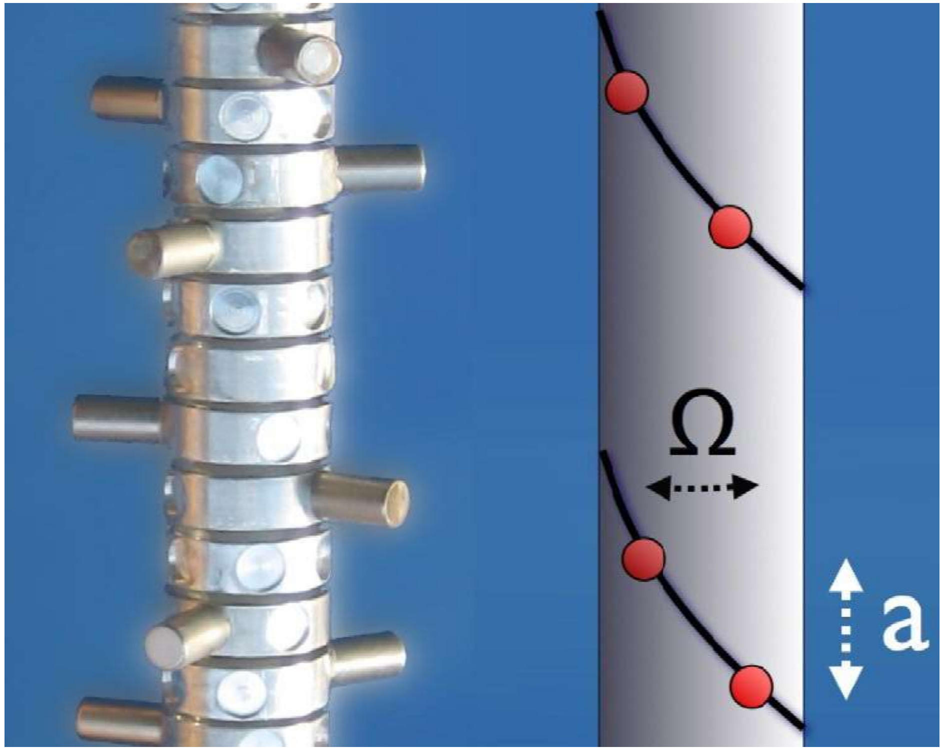
$$x = \frac{\alpha}{2\pi},\ y = \frac{h}{2\pi} \quad r_{n,m} = \left(\frac{m - nx}{\sqrt{y}}, n\sqrt{y} \right)$$

$$U(x,y) \sim \sum_{\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}} e^{-\beta \left(\frac{(n-nx)^2}{y} + yn^2 \right)}$$



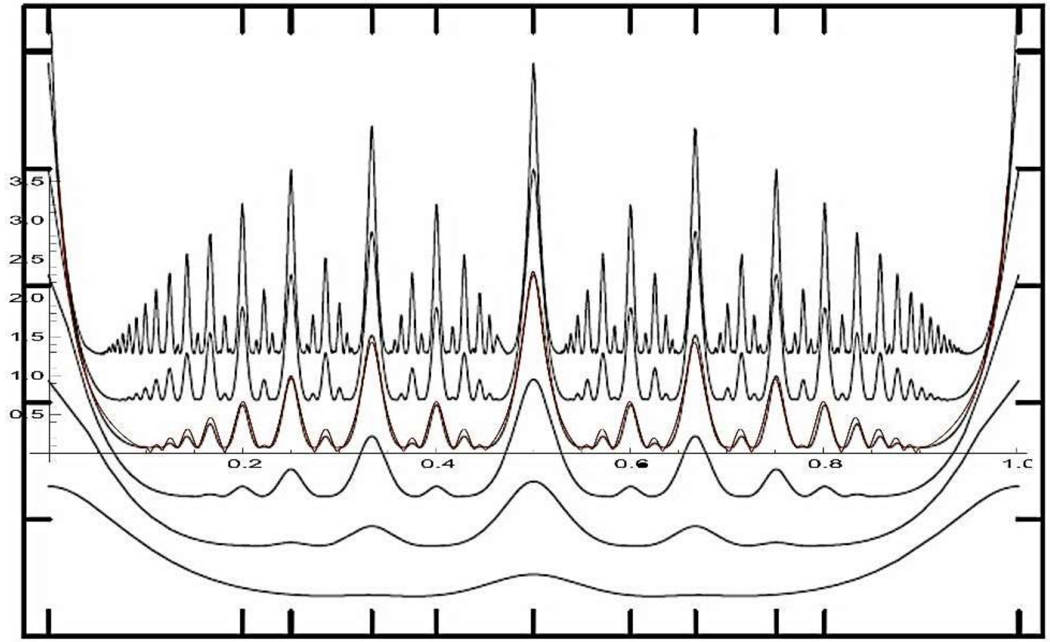


Static and Dynamical Phyllotaxis in Magnetic Cactus
C. Nisoli et al, ArXiv: cond-mat/0702335



Experimental setting

Hierarchical potential energy relief between states with various Ω



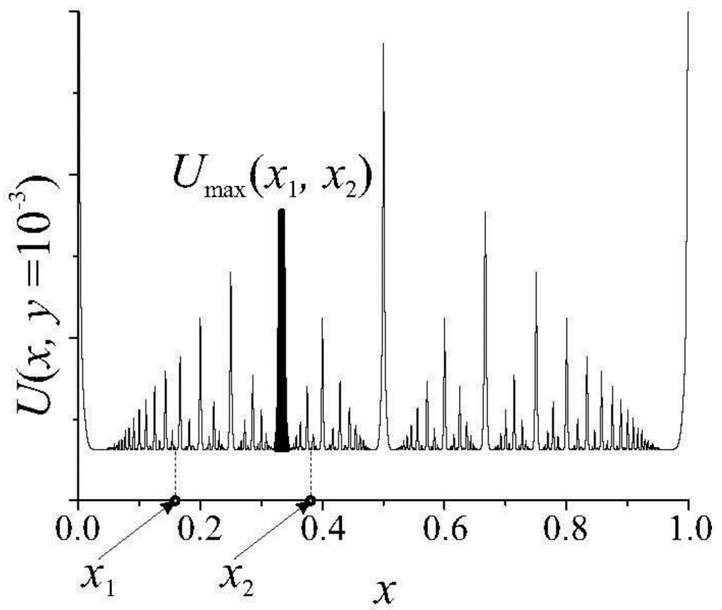
Definition of the ultrametric space

The pairwise distance, $d(x_1, x_2)$ between elements x_1 and x_2 is *ultrametric* if it meets three requirements:

- It is non-negative
  $d(x_1, x_2) > 0$ for $x_1 \neq x_2$ and $d(x_1, x_2) = 0$ for $x_1 = x_2$
- It is symmetric
  $d(x_1, x_2) = d(x_2, x_1)$
- It obeys *the strong triangle inequality*,
  $d(x_1, x_2) < \max\{d(x_1, x_3), d(x_3, x_2)\}$

  instead of the ordinary triangle inequality typical for Euclidean spaces, $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$

We can identify the energy landscape with a metric space



$$U(x,y) \sim \sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} e^{-\beta\left(\frac{(m-nx)^2}{y}+yn^2\right)}$$

Considering the profile $U(x, y)$ as a function of x, we set $d(x_1, x_2) = U_{max}(x_1, x_2)$ where $d(x_1, x_2)$ is the ultrametric pairwise distance

At $\beta >> 1$ one can approximately rewrite $U(x, y)$ as

$$U(x,y)\Big|_{\beta\gg 1} \approx \sqrt{\frac{\pi}{\beta}} \sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} \delta\left(\frac{(m-nx)^2}{y}+yn^2\right)$$

Making use of regularization of the δ-function, we represent the potential $U(x, y)$ at $\beta >> 1$ as follows:

$$U(x,y)\Big|_{\beta\gg 1} = \lim_{\beta\to\infty}\sqrt{\frac{\beta}{\pi}} \sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} \frac{1}{\beta\left(\frac{(m-nx)^2}{y}+yn^2\right)+\frac{1}{\beta}}$$

$$\approx \frac{1}{\sqrt{\pi\beta}} \sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} \frac{1}{Q^s(m,n)}$$

where $s \to 1$ and $Q(m, n)$ is a positive quadratic form

$$Q(m,n) = \frac{1}{y}m^2 - \frac{2x}{y}mn + \left(\frac{x^2}{y}+y\right)n^2$$

Recall now the definition of the Eisenstein series

$$E(\tilde{z},s) = \frac{1}{2} \sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} \frac{\tilde{y}^s}{|m\tilde{z}+n|^{2s}}; \qquad \tilde{z} = \tilde{x} + i\tilde{y}$$

Reminder of Eisenstein series

$E(z, s)$ as a function of z, is a $SL(2,Z)$–invariant automorphic solution of the hyperbolic Laplace equation

$$-\tilde{y}^2\left(\frac{\partial^2}{\partial\tilde{x}^2}+\frac{\partial^2}{\partial\tilde{y}^2}\right)E(\tilde{x},\tilde{y},s)=s(1-s)\;E(\tilde{x},\tilde{y},s)$$

$E(z, s)$ is related to the Epstein ζ-function, defined as

$$\zeta(Q,s)=\sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}}\frac{1}{Q^s(m,n,\tilde{z})}=\frac{2}{d^{s/2}}E(\tilde{z},s)$$

where

$$Q(m,n,\tilde{z})=a(\tilde{z})m^2+2b(\tilde{z})mn+c(\tilde{z})n^2$$

Comparing $Q(m,n,z)$ to $Q(m,n)=\frac{1}{y}m^2-\frac{2x}{y}mn+\left(\frac{x^2}{y}+y\right)n^2$, we can make identifications

$$\frac{1}{y}=a,\quad \frac{x}{y}=-b,\quad \frac{x^2}{y}+y=c \qquad d=ac-b^2=1$$

$$z=\frac{b+i\sqrt{d}}{a} \quad \Rightarrow \quad \left(\tilde{x}=\frac{b}{a} \qquad \tilde{y}=\frac{\sqrt{d}}{a}=\frac{1}{a}\right)$$

Kronecker 1st limit formula

The behavior of $\zeta(Q, s)$ at $s \to 1$ is known as 1st Kronecker limit formula

$$\zeta(Q,s) = \frac{\pi}{\sqrt{d}}\frac{1}{s-1} + \frac{2\pi}{\sqrt{d}}\left(\gamma + \ln\sqrt{\frac{a}{4d}} - 2\ln|\eta(z)|\right) + O(s-1)$$

where $\eta(z) = e^{\pi i z/12}\prod_{n=0}^{\infty}(1 - e^{2\pi i n z})$ is the Dedekind eta-function

$$\eta(z+1) = e^{\pi i z/12}\,\eta(z) \qquad \eta\left(-\frac{1}{z}\right) = \sqrt{-i}\,\eta(z)$$

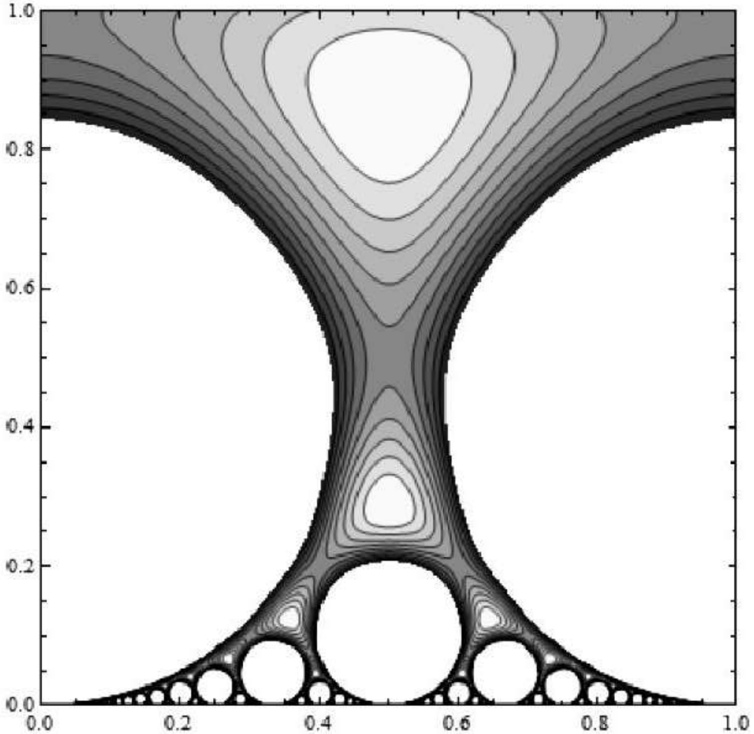
Dropping the divergent at $s \to 1$ term, we get for the ultrametric potential $U(x, y)$:

$$U(x,y) \approx \sqrt{\frac{\pi}{\beta}}\left(\frac{\pi}{s-1} + 2\pi\gamma - \ln 2 - 4\pi\ln h(x,y)\right)$$

where $h(x,y) = y^{1/4}|\eta(x+iy)|$

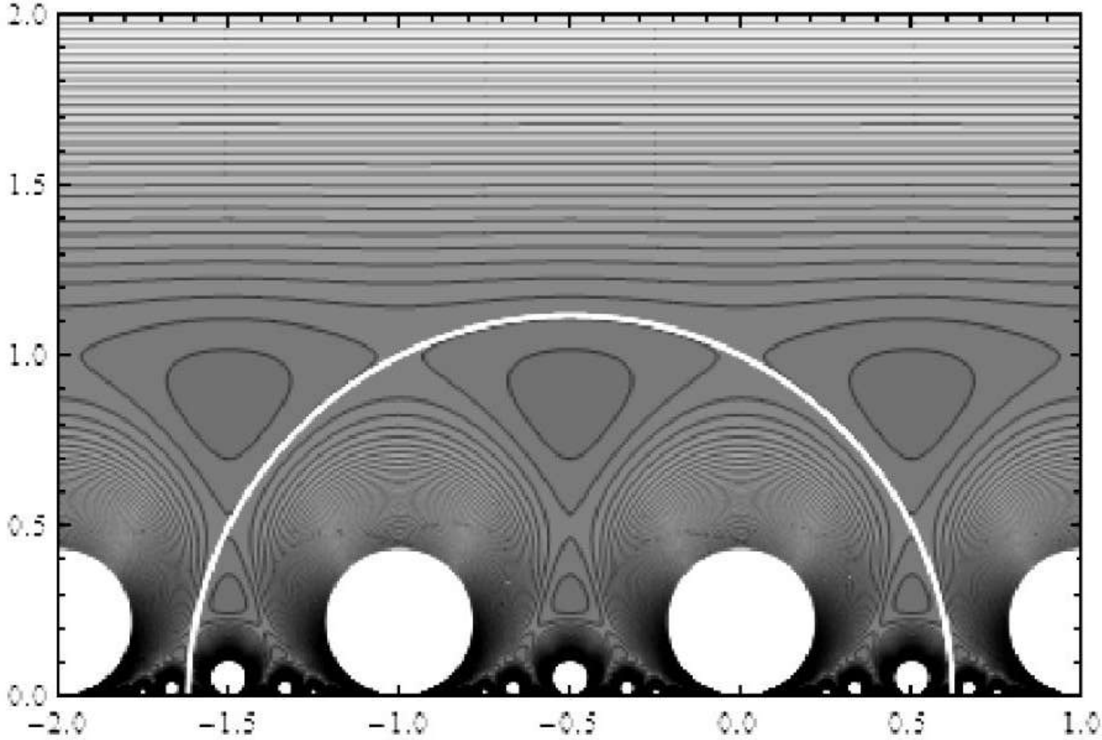
Relief of the function

$$h(x,y) = y^{1/4}|\eta(x+iy)|$$



Relief of the potential

$$\tilde{U}(x,y) = -\ln h(x,y)$$



The *longest open geodesic* is the largest horocycle defined by the equation

$$\left(x+\frac{1}{2}\right)^2 + y^2 = \frac{5}{4}$$

To proceed with regularization of a Riemann function, return to the ”phyllotaxis potential” and assess $U(x, y)$ at small $y \to 0$:

$$U(x, y \to 0)\Big|_{\beta \gg 1} \approx \frac{1}{y\sqrt{\beta\pi}} \sum_{\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{y^2}{(m+nx)^2 + y^2 n^2}$$

$$\approx \frac{2}{y\sqrt{\beta\pi}} \sum_{m=1} \sum_{n=1} \frac{1}{n^2} \delta\left(x - \frac{m}{n}\right)$$

Comparing with the Riemann “raindrop” function, we get

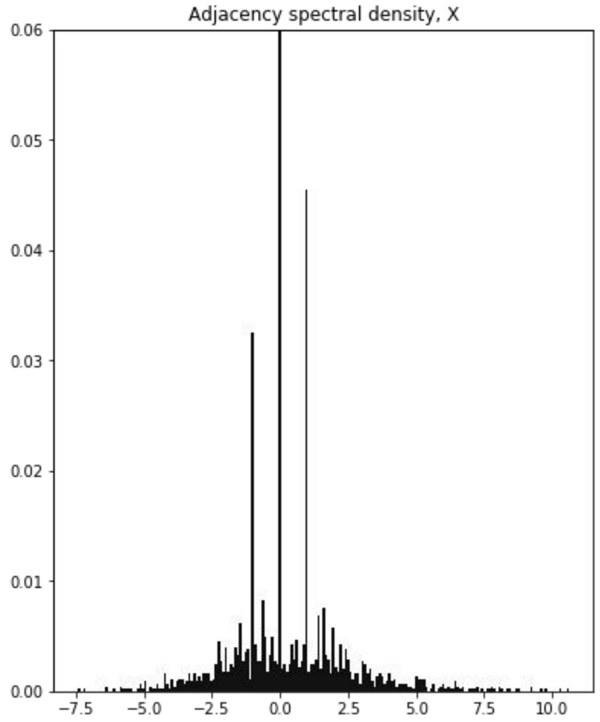
$$\frac{\pi}{12y} g^2(x) = -\ln|\eta(x+iy)|_{y\to 0} + O(\ln y)$$

and, finally

$$g(x) \to \sqrt{-\frac{12y}{\pi} \ln|\eta(x+iy)|}\,\Bigg|_{y\to 0}$$

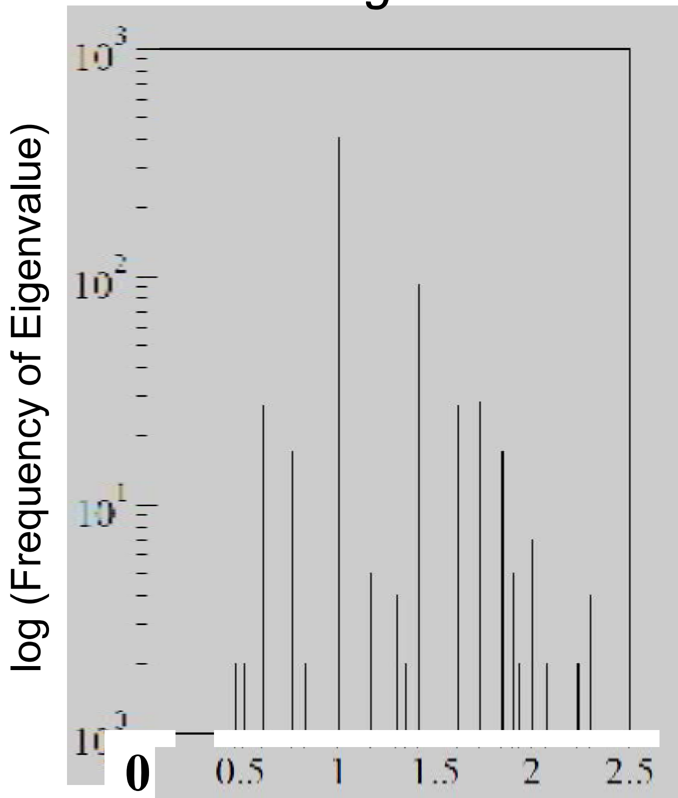
Samples of eigenvalue densities of sparse networks of different physical/biological nature

Typical spectral statistics of adjacency matrix of *X chromosome*



eigenvalue

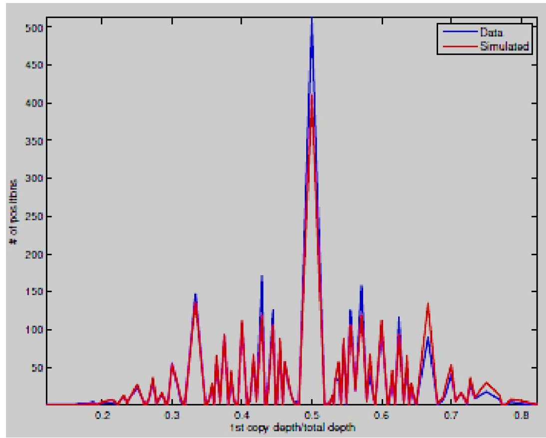
Spectral statistics of protein-protein interaction network in *Drosophyla melanogaster*



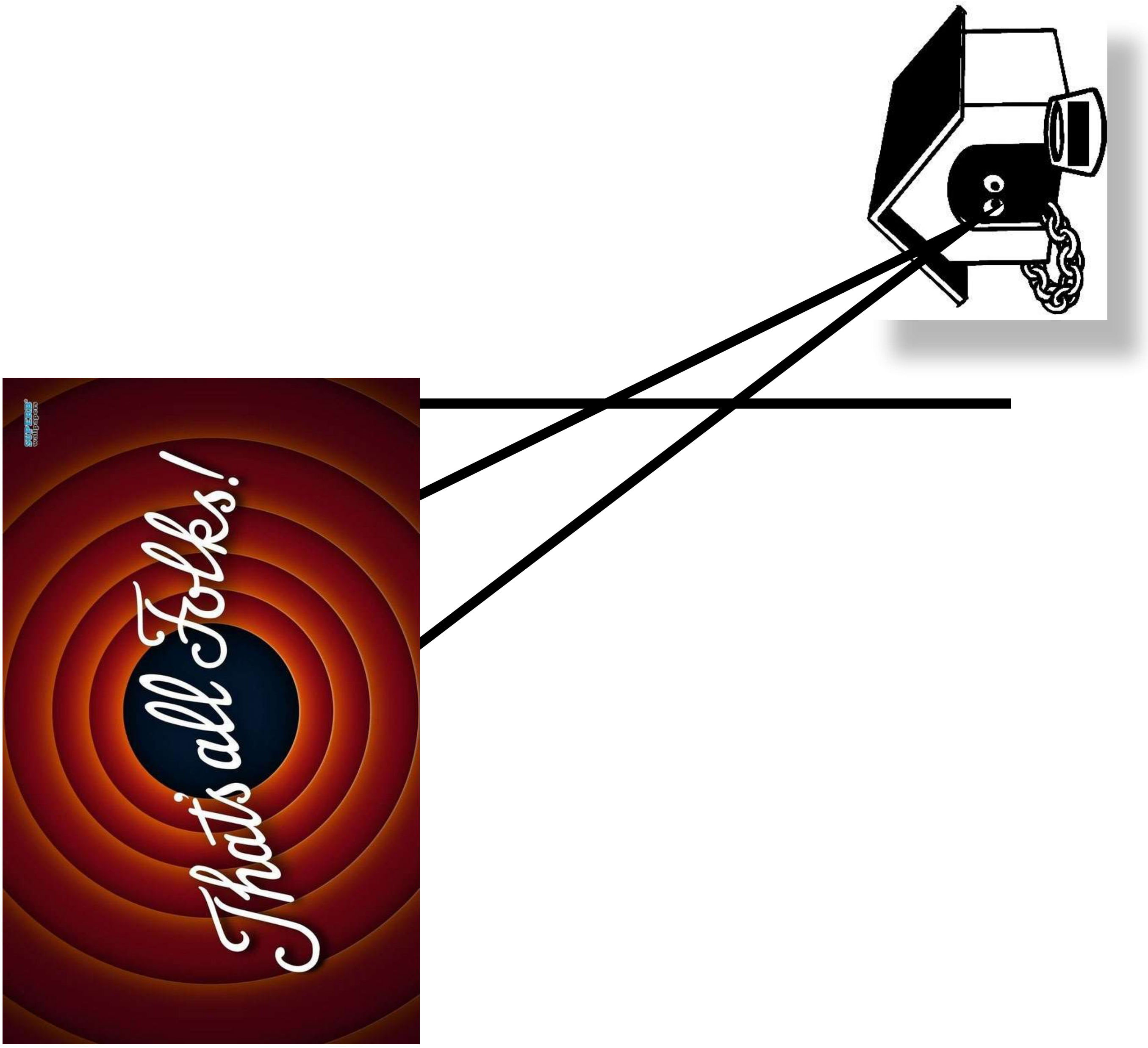
eigenvalue

C. Kamp, K. Christensen (Phys.Rev E, 2005)

Distributions of ratios from heterozygous single nucleotide polymorphisms data from the *sequencing of a cancer genome*



V. Trifonov, L. Pasqualucch, R. Dalla-Favera, R.Rabadan (Sci. Rep, 2011)


That's all Folks!